Crossing the brachistochrone

Background image: Milky Way over the Pinnacles in Australia

[https://apod.nasa.gov/apod/ap160217.html]

Prologue

Perhaps the most famous equation in all physics is Albert Einstein's formula

$$E = mc^2$$

Specifically, because the relativistic mass m changes, it is

$$E_0 = m_0c^2,$$

$$E = \gamma_L m_0c^2 = mc^2$$

where m_0 is the 'rest,' or inertial, mass.

The relativistic factor γ_L is called Lorentz factor and its value is

$$\gamma_L = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}$$

The problem here is that if the speed v (of some object) approaches the speed of light c, Lorentz factor goes to infinity, so that the relativistic energy will also be infinite,

$$E(v) = \gamma_L E_0 = \gamma_L m_0c^2, \qquad E_0 = m_0c^2, \qquad v = 0, \qquad \gamma_L(v) = 1$$

$$E(v) = \infty, \qquad v = c, \qquad \gamma_L(v) = \infty$$

A spontaneous thought of mine was to suppose for the energy a formula of the form

$$E(v) \approx E_0e^{-\gamma_L}$$

Later on I discovered that this energy had better be

$$E(t) = E_0e^{-\gamma t},$$

$$E(n) = E_0e^{-n}, \qquad \gamma t = n,$$

$$E(v) = E_0e^{-\frac{v}{c}}, \qquad n = \frac{v}{c} \Rightarrow$$

$$E(v) = E_0e^{-\beta_L}, \qquad \gamma_L = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{1}{1 - \beta_L^2}}, \qquad \beta_L = \sqrt{1 - \frac{1}{\gamma_L^2}} = \frac{v}{c}$$

where γ is the damping factor (distinct from γ_L).

The comparison between the two factors made me think that motion in spacetime is wave-like, so that spacetime is fundamentally an oscillatory medium.

I found out that a wave can be approximated with a brachistochrone, and I wondered what the connection between motion in spacetime and motion on the brachistochrone could be.

Since the brachistochrone is the path of least action, I assumed that all (accelerated) motions occur on brachistochrones.

Then I made the connection between the energy of the brachistochrone and that of an oscillator, so that motion in spacetime can also be described as motion on the brachistochrone.

I also devised a thought experiment (the spaceship on the brachistochrone) in order to relate an object moving on the brachistochrone to the photons the object emits, and I came up with a condition of simultaneity,

$$v\tau = cT_0$$

where v is the speed of the object, τ is the period of the photons, c is the speed of photons (the speed of light), and T_0 is the time of the brachistochrone.

The intimate relationship between photons and objects (or material particles) can also be described in the context of wave- particle duality, using the following equations

$$p_m = mv$$

$$p_\mu = \mu c$$

where p_m and p_μ are the momenta of an object (a particle) and of a wave (a photon), respectively.

Significant is here that the mass m of the object is different from the mass μ of the wave. The deepest meaning is that photons are disturbances in spacetime caused by the moving object. A consequence is that an object can exceed the speed of light.

Then came to my mind the idea of synchronicity as put forward by Carl Jung. According to his description, there is a correspondence between physical and psychic phenomena based on the action of archetypes. Thus I wondered what the relationship between spacetime and Consciousness could be.

At that moment I recalled that in the energy equation of the oscillator, a constant $\mathcal{C}$ of integration appears,

$E(t) = E_0 e^{-\gamma t} \Rightarrow$

$\ln E(t) = \ln E_0 - \gamma t = \mathcal{C} - \gamma t,$

$E_0 = e^{\mathcal{C}}$

I identified the constant $\mathcal{C}$ with Consciousness.

The correspondence between physical phenomena and observation can also be revealed by numerical coincidences between the values of fundamental quantities and according to the anthropic principle.

I also formulated a set of rules, based on the aforementioned condition of simultaneity, including a principle of synchronicity.

From there after I made some thoughts with respect to the notion of information and the holographic principle, so that I came up with the following equation (which appears inscribed in the previous picture),

$\mathcal{C} = \ln \Gamma(t) + I(t),$

$\mathcal{C} = \ln E_0$

$\Gamma(t) = e^{\gamma t},$

$I(t) = \ln E(t)$

where $I(t)$ is the available information, and $\Gamma(t)$ is what I call factor of Free Will.

Explicitly the previous equation can be written as

$$\mathcal{C} = \ln E(t) + \gamma t = \ln E_0\,, \qquad \gamma t = \frac{v}{c} = n, \qquad n = [1,2,\ldots]$$

where the state or harmonic n also represents the degrees of freedom available to Free Will,

$$\ln \Gamma(t) = n$$

Thus Consciousness can be seen as a material object which transforms the total available energy E_0 into its own kinetic energy $K(t)$,

$$E(t) = E_0 e^{-\gamma t} \Rightarrow$$

$$K(t) = \Delta E(t) = E_0 - E(t) = E_0 - E_0 e^{-\gamma t} = E_0(1 - e^{-\gamma t})$$

acquiring knowledge in a process which I call displacement of Consciousness,

$$\ln K(t) = \ln \Delta E(t) = \ln E_0 + \ln\left[1 - \frac{1}{\Gamma(t)}\right]$$

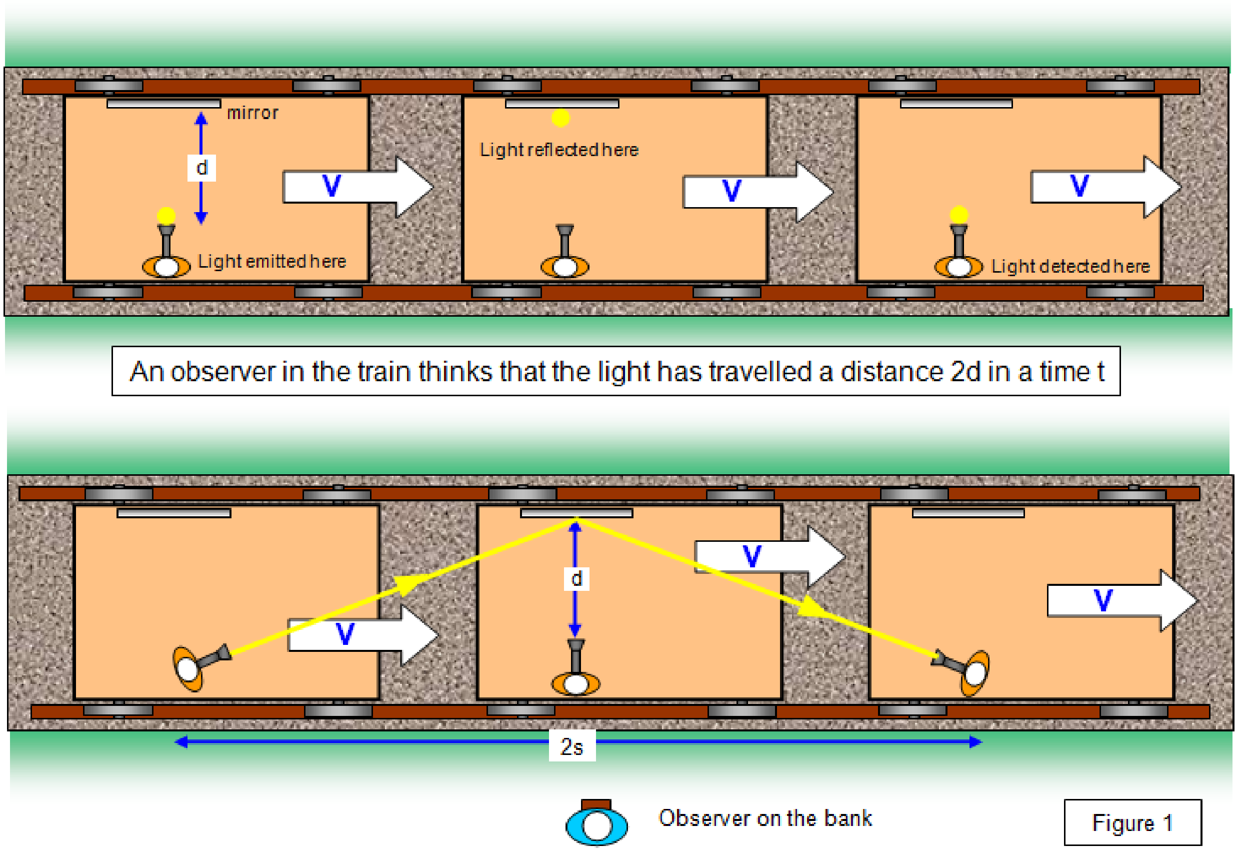
This way one may integrate the observer with the phenomena he/she observes. In string theory for example each physical property corresponds to the mode or harmonic of a vibrating string, so that a physical object is the collection of those harmonics n.

If we identify such strings with archetypes then each harmonic or state can be related both to a physical property and to a psychic property (emotion).

Consequently Consciousness will be the sum of all the states or harmonics n.

Part 1

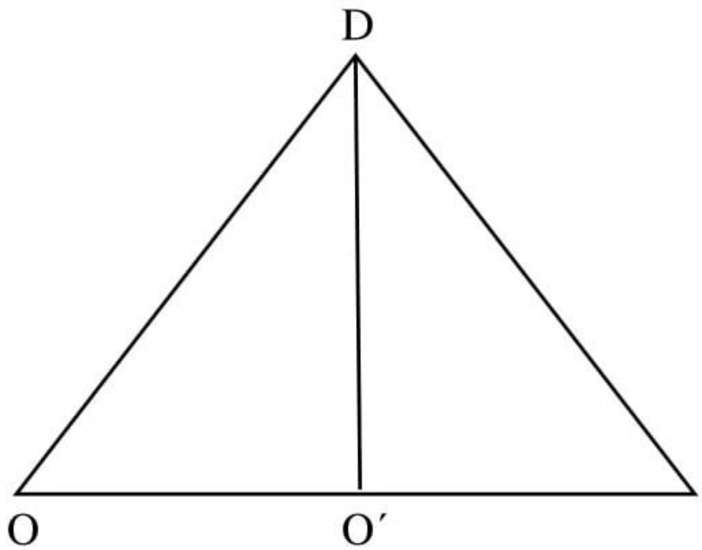
Einstein's train



[http://www.schoolphysics.co.uk/age16-19/Relativity/text/Time_dilation/index.html]

In Einstein's original thought experiment on relativity, a passenger (Alice) is on a train. As she moves, she sends a beam of light (a photon) at a mirror on the roof of the train. Because Alice stands still with respect to the train, the photon is reflected vertically back to Alice. On the other hand, Bob, an observer outside the train, sees the light signals displaced because of the motion of the train, relative to him. Thus the distance the photon travels is different for the two observers. But because the speed of light is constant, the time elapsed for the two observers will also be different.

The problem can be illustrated with the following triangle:



If the vertical distance between Alice (at point *O΄*) and the mirror is *O΄D*, and it takes some time *Δt΄*, according to Alice, for the light to travel back and forth this distance, then it will be *O΄D=cΔt΄/2*, where *c* is the speed of light. On the other hand, if *v* is the speed of the train, then at some time *Δt*, according to Bob (at point *O*), the train will have moved a distance *OO΄=vΔt/2*, while the light signals will have travelled a distance *OD=cΔt/2*. It is supposed that at the moment Alice emitted the signal, the train passed in front of Bob (so that originally the points *O* and *O΄* coincided).

The relationship between the times, according to the two different observers, is given by applying the Pythagorean theorem on the previous triangle:

$$(OD)^2 = (OO')^2 + (O'D)^2 \Rightarrow$$

$$\left(\frac{c\Delta t}{2}\right)^2 = \left(\frac{v\Delta t}{2}\right)^2 + \left(\frac{c\Delta t'}{2}\right)^2 \Rightarrow$$

$$c^2(\Delta t)^2 = v^2(\Delta t)^2 + c^2(\Delta t')^2 \Rightarrow$$

$$c^2(\Delta t')^2 = c^2(\Delta t)^2 - v^2(\Delta t)^2 = (c^2 - v^2)(\Delta t)^2 \Rightarrow$$

$$(\Delta t)^2 = \frac{c^2}{c^2 - v^2}(\Delta t')^2 \Rightarrow$$

$$\Delta t = \sqrt{\frac{c^2}{c^2 - v^2}}\Delta t' = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}\Delta t \Rightarrow$$

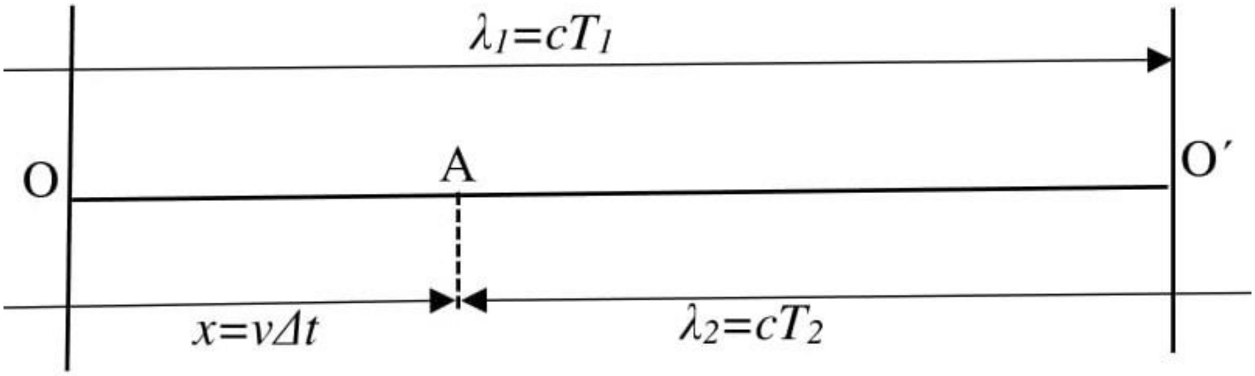
$$\Delta t = \gamma_L \Delta t', \qquad \gamma_L = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}$$

The relativistic factor γ_L is Lorentz factor. This factor goes from *1* to infinity, as the speed v goes from *0* to the speed of light c. Thus γ_L can be defined only if $v<c$.

The corridor example

A way to treat the previous problem, that Lorentz factor goes to infinity if an object moves at the speed of light, is to make the distinction between the coordinate time (the time passing on the clocks of observers), and the period of the photons (the time 'on the clock' of the photons).

This is a related sketch:



Here we have the same observer (instead of two), moving on the horizontal axis $x=(OO')$. The observer uses photons to estimate the distance he/she travels. Initially the observer is at rest, at point O. The photon is reflected at point O', and returns to the observer when he/she is at point A. If it originally takes the photon a time T_1 to travel the distance OO', this distance will be $OO'=cT_1$. If the observer travels at a speed v, then, after some time Δt, according to his/her own clock, he/she will have travelled a distance $OA=v\Delta t$. During the same time Δt, the photon will have travelled a total distance $OO'+O'A$, where $O'A=cT_2$, till the observer takes back the photon at point A.

Presumably, the times T_1 and T_2 refer to the period of the photon, so that the lengths $\lambda_1=cT_1$ and $\lambda_2=cT_2$ will be wavelengths. The distance the observer travels can also be called Δx, so that we have,

$(OO') \equiv \lambda_1 = cT_1$

$(O'A) \equiv \lambda_2 = cT_2$

$(OA) \equiv \Delta x = v\Delta t$

The time Δt it takes the observer to reach point A, will be equal to the time T_1 it takes the photon to travel from point O to point O', plus the time T_2 it takes the photon to reach point A from point O',

$\Delta t = t_2 - t_1 = T_1 + T_2,$

$\Delta T = T_1 - T_2$

Thus for the distances we have,

$(OO') = (OA) + (AO') \Rightarrow$

$cT_1 = v\Delta t + cT_2 \Rightarrow$

$v\Delta t = cT_1 - cT_2 = c(T_1 - T_2) = c\Delta T \Rightarrow$

$v(T_1 + T_2) = c(T_1 - T_2)$

This formula relates the displacement $\Delta x = v\Delta t$ of the observer, to the displacement $\Delta\lambda = c\Delta T$ of the reference (the emitted) photon.

The fundamental aspect here is that, while the time Δt refers to the clock of the observer, the times T_1 and T_2 refer to the period of the photons.

The time t, as measured by the clock of the moving observer, although still a 'related' time, it is not a 'relative' time, because it does not refer to an external observer, but it is intimately connected to the period T of the photon itself.

On the other hand, the period of the photon depends on the motion of the observer. In other words, the photon changes its period, because of the motion of the observer,

$cT_1 = v\Delta t + cT_2 \Rightarrow$

$cT_1 = v(T_1 + T_2) + cT_2 \Rightarrow$

$cT_1 = vT_1 + vT_2 + cT_2 \Rightarrow$

$cT_1 - vT_1 = vT_2 + cT_2 \Rightarrow$

$(c - v)T_1 = (v + c)T_2 \Rightarrow$

$$\frac{T_2}{T_1} = \frac{c - v}{c + v}$$

The dependence of the photon's period on the speed of the observer, also tells us that the motion (i.e. the speed) of the observer cannot be defined independently of the photon,

$v\Delta t = c\Delta T \Rightarrow$

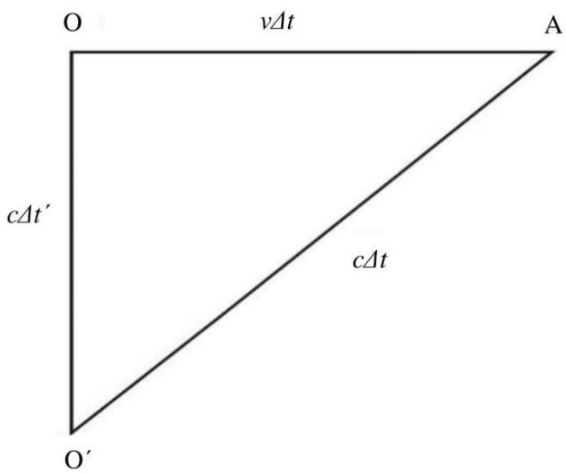
$$v = c\frac{\Delta T}{\Delta t} = c\frac{T_1 - T_2}{t_2 - t_1} = c\frac{T_1 - T_2}{T_1 + T_2}$$

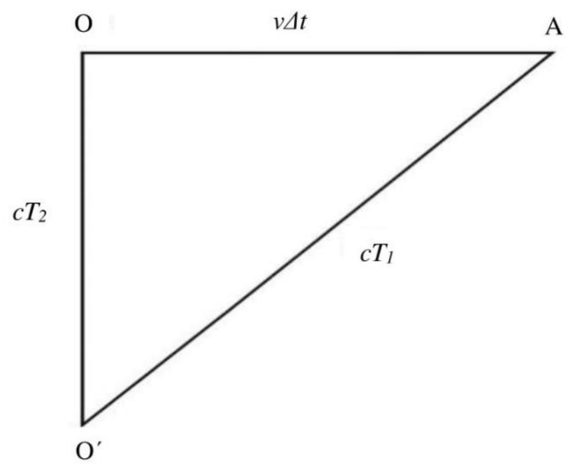
Thus the motion of the observer is intimately related to the aspects of the photon. However this relationship involves the photon and the same observer. Therefore the motion is not relative (it does not refer to an external observer), but absolute (it refers to the photon as the medium in which motion occurs).

The most profound interpretation of such an effect is that the photon is not a 'point (a particle) travelling in empty spacetime,' but that in fact it is a disturbance of spacetime, which in turn is caused by the motion of the observer. Such an aspect will be significant in our discussion.

A relativistic triangle

The corridor example can be expanded in two dimensions, so that the corridor can be replaced by a triangle:





The first triangle on the left, corresponds to Einstein's train example. Instead of a train, we can imagine an airplane, moving from point O to point A, emitting a photon towards point $O´$, and taking back the photon at point A. Equivalently the description can be made with respect to an observer at rest, at point $O´$, who emits a photon, and traces the airplane when the latter reaches point A. In such a sense the time $\Delta t´$ is related to a clock on the airplane (proper time), while the time Δt is related to the clock of the observer at rest, at point $O´$ (coordinate time).

Thus, with respect to the first triangle in the previous image, we have

$(O'A) = c\Delta t$

$(OO') = c\Delta t'$

$(OA) = \Delta x = v\Delta t$

$(O'A)^2 = (OO')^2 + (OA)^2 \Rightarrow$

$c^2(\Delta t)^2 = c^2(\Delta t')^2 + v^2(\Delta t)^2 \Rightarrow$

$c^2(\Delta t')^2 = c^2(\Delta t)^2 - v^2(\Delta t)^2 = (c^2 - v^2)(\Delta t)^2 \Rightarrow$

$$(\Delta t')^2 = \frac{c^2 - v^2}{c^2}(\Delta t)^2 \Rightarrow$$

$$\Delta t' = \sqrt{\frac{c^2 - v^2}{c^2}} \Delta t \Rightarrow$$

$$\Delta t' = \frac{1}{\gamma_L} \Delta t, \qquad \gamma_L = \sqrt{\frac{c^2}{c^2 - v^2}} = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}$$

which gives us the result we have already seen in Einstein's train example.

The second triangle on the right, is the two- dimensional analogue of the corridor example. In this case the airplane, as it leaves point *O*, emits a photon towards point *O΄*, and takes back the photon at point *A*. If T_1 is the initial period of the photon at point *O* (before the plane starts to move), and T_2 is the photon's period when the airplane reaches point *A*, then we have

$$(O'A) = \lambda_1 = cT_1$$
$$(OO') = \lambda_2 = cT_2$$
$$(OA) = \Delta x = v\Delta t$$
$$\Delta t = t_2 - t_1 = T_1 + T_2$$
$$(O'A)^2 = (OO')^2 + (OA)^2 \Rightarrow$$
$$c^2T_1^2 = c^2T_2^2 + v^2(\Delta t)^2$$

If the last equation is written in the following form,

$$\lambda_1^2 = \lambda_2^2 + v^2(\Delta t)^2 \Rightarrow$$
$$\lambda_2^2 = \lambda_1^2 - v^2(\Delta t)^2$$

it can be associated with the notion of length contraction in relativity.

However the interpretation here is fundamentally different. This is because the length contraction is not due to relative motion, but to the effects motion has on the photon, or on spacetime (if we treat photons as fluctuations of spacetime). Such effects will ultimately lead us to the principle of synchronicity, later on.

Notes:

There are two ways to treat the previous expression

$$c^2T_2^2 = c^2T_1^2 - v^2(\Delta t)^2$$

One way is to associate the time Δt, which is the time the observer measures on his/her clock, with the initial period T_1 of the reference photon (the period which the photon has before the observer starts to move). Thus setting,

$$\Delta t \equiv T_1$$

we take,

$$c^2T_2^2 = c^2T_1^2 - v^2\Delta t^2 \Rightarrow$$

$$c^2T_2^2 = c^2T_1^2 - v^2T_1^2 = (c^2 - v^2)T_1^2 \Rightarrow$$

$$T_2^2 = \frac{c^2 - v^2}{c^2}T_1^2 \Rightarrow$$

$$T_2 = \sqrt{\frac{c^2 - v^2}{c^2}}T_1 = \sqrt{1 - \frac{v^2}{c^2}}T_1$$

Identifying now,

$$T_1 \equiv \Delta t, \qquad T_2 = \Delta t'$$

we have that,

$$\Delta t' = \sqrt{1 - \frac{v^2}{c^2}}\Delta t$$

Thus we take back the common relativistic expression.

On the other hand, if we associate the time Δt, with the final period T_2 of the reference photon,

$$\Delta t \equiv T_2$$

then we take,

$$c^2T_2^2 = c^2T_1^2 - v^2\Delta t^2 \Rightarrow$$
$$c^2T_2^2 = c^2T_1^2 - v^2T_2^2 \Rightarrow$$
$$c^2T_2^2 + v^2T_2^2 = (c^2 + v^2)T_2^2 = c^2T_1^2 \Rightarrow$$
$$T_2^2 = \frac{c^2}{c^2 + v^2}T_1^2 \Rightarrow$$
$$T_2 = \sqrt{\frac{c^2}{c^2 + v^2}}T_1 = \sqrt{\frac{1}{1 + \frac{v^2}{c^2}}}T_1$$

This formula, with a positive sign in the square root, in fact gives us back the relativistic expression, as a linear approximation at small speeds.

The linear approximation of a function of the form $(1+x)^n$, is given as

$$(1 + x)^n \approx 1 + nx, \qquad x \ll 1$$

so that, setting $x=v^2/c^2$, it will be

$$\left(1 + \frac{v^2}{c^2}\right)^n \approx 1 + n\frac{v^2}{c^2}, \qquad x = \frac{v^2}{c^2}, \qquad x \ll 1, \qquad v \ll c$$

Thus, defining

$$\gamma_L^+ = \sqrt{\frac{c^2}{c^2 + v^2}} = \sqrt{\frac{1}{1 + \frac{v^2}{c^2}}}, \qquad \gamma_L^- = \sqrt{\frac{c^2}{c^2 - v^2}} = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}$$

where γ_L^- is the common Lorentz factor,

and supposing that the speed v is sufficiently small, we take

$$v \ll c \Rightarrow$$
$$\gamma_L^+ = \sqrt{\frac{1}{1 + \frac{v^2}{c^2}}} = \left(1 + \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}\frac{v^2}{c^2} \approx \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = \frac{1}{\gamma_L^-}$$

Therefore the formula

$$T_2 = \sqrt{\frac{c^2}{c^2 + v^2}} T_1 = \sqrt{\frac{1}{1 + \frac{v^2}{c^2}}} T_1 \approx \sqrt{1 - \frac{v^2}{c^2}} T_1$$

will correspond to the relativistic expression

$$\Delta t' = \sqrt{1 - \frac{v^2}{c^2}} \Delta t$$

at the limit of small speeds, $v << c$.

The significance is that the times t' and t, as measured by the clocks of observers (either moving, or at rest, respectively), can be intimately connected to the periods (T_2 and T_1, respectively) of the photon the observers use to make measurements.

This intricate relationship also reveals the deepest aspect that photons are oscillations in spacetime (if the oscillations are measured in the form of photons), so that the reduction of the period of the photons corresponds to the reduced period of the oscillating spacetime.

This period goes to zero, only if the speed of the moving observer becomes infinite, as it is suggested by the formula

$$T_2 = \sqrt{\frac{1}{1 + \frac{v^2}{c^2}}} T_1,$$

$$v \gg c \Rightarrow$$

$$T_2 = \sqrt{\frac{1}{1 + \frac{v^2}{c^2}}} T_1 \approx \sqrt{\frac{1}{\frac{v^2}{c^2}}} T_1 = \frac{c}{v} T_1 = \frac{1}{n} T_1, \qquad n = \frac{v}{c}$$

where the altered period T_2, becomes a submultiple n of the initial period T_1.

As a consequence, faster than light travel is possible.

Doppler shift

A common way by which we experience the change of frequency (thus also the period) of a wave, is, for example, the sound of an ambulance's siren, as it passes by. As the sound wave is deformed, the pitch goes higher as the ambulance approaches, and it goes lower as the ambulance recedes.

This is the Doppler shift for sound, and the frequency is given as follows. If f_E is the frequency of the emitter (the moving ambulance), f_R is the frequency of the receiver (an observer close by), v_E is the speed of the ambulance, v_R is the speed of the observer (assuming that the observer is also in motion), c is the speed of the sound wave (here it is not the speed of light), and λ is the wavelength of the sound wave, then we have

$$f_R = \frac{c \pm v_R}{\lambda_R} = \frac{c \pm v_R}{c \pm v_E} f_E,$$

$$\lambda_R = \frac{c \pm v_E}{f_E}$$

where the plus or minus sign depends on whether the source and the receiver approach or recede one from the other.

This formula implies that the two speeds, v_E and v_R, of the emitter and of the receiver, respectively, are calculated with respect to a fixed point in the medium (the air in this case), so that the medium can be considered as the absolute frame of reference to which both observers refer. However if we treat the source (the emitter) as stationary, and suppose that the observer (the receiver) moves with respect to the source, so that we set $v_E=0$ and $v_R=v$, the previous formula takes the form

$$f_R = \frac{c \pm v}{\lambda_R} = \frac{c \pm v}{c} f_E,$$

$$\lambda_R = \frac{c}{f_E}$$

This equivalent expression contains the relative speed v of the moving observer (the receiver) with respect to the emitter (the source), so that the medium is ignored.

From this formula we can reach the expression for the (relativistic) Doppler shift for light, if we additionally suppose that the wavelength λ_R of the light wave, as perceived by the receiver (the observer moving at the speed v), is contracted by the Lorentz factor γ_L, so that the calculation is as follows,

$$f_R = \frac{c \pm v}{\lambda_R}, \qquad v_E = 0, \qquad v_R = v \Rightarrow$$

$$f_R = \frac{c \pm v}{\lambda_R} = \gamma_L \frac{c \pm v}{c} f_E, \qquad \lambda_R = \frac{1}{\gamma_L}\lambda_E = \frac{1}{\gamma_L}\frac{c}{f_E}, \quad \lambda_E = \frac{c}{f_E} \Rightarrow$$

$$f_R = \sqrt{\frac{c^2}{c^2 - v^2}}\frac{c \pm v}{c} f_E, \qquad \gamma_L = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{c^2}{c^2 - v^2}} \Rightarrow$$

$$f_R = \sqrt{\frac{c^2}{c^2 - v^2}\left(\frac{c \pm v}{c}\right)^2} f_E = \sqrt{\frac{c^2}{c^2 - v^2}\frac{(c \pm v)^2}{c^2}} f_E = \sqrt{\frac{(c \pm v)^2}{c^2 - v^2}} f_E \Rightarrow$$

$$f_R = \sqrt{\frac{(c \pm v)^2}{(c + v)(c - v)}} f_E = \sqrt{\frac{c \pm v}{c \mp v}} f_E$$

where here c is the speed of light, while the expression

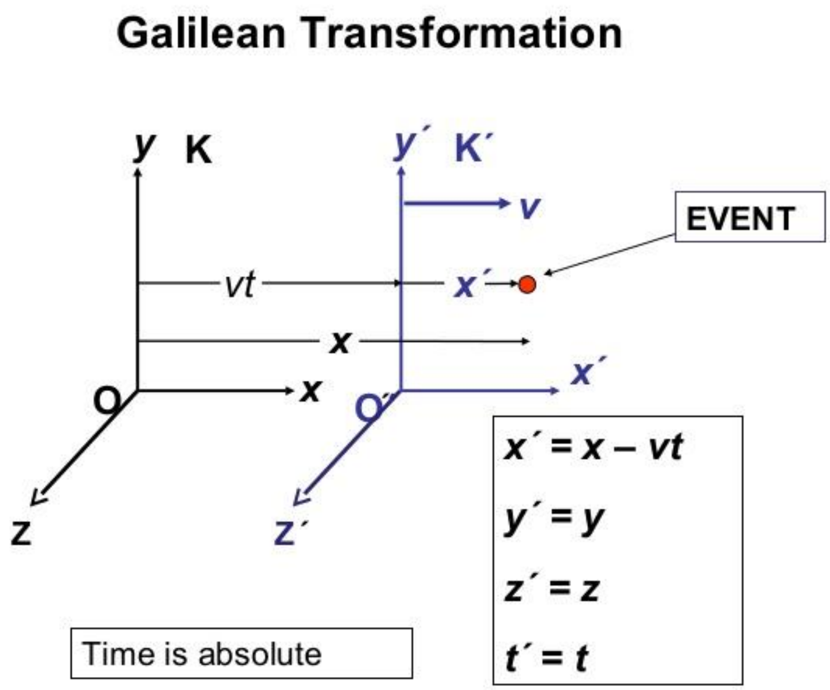
$$\lambda_R = \frac{1}{\gamma_L}\lambda_E$$

is based on Lorentz transformation for length contraction in relativity.

We may notice that although the previous relationship for the change of frequency was derived by assuming that the speed v of the moving observer is relative to a source (the emitter) standing still, the same speed may also be related to a fixed point in the medium (the wave itself), independently of any external source or observer.

Lorentz transformation

Here we can show that Doppler shift for the frequency (thus also for the period) can be derived from Lorentz transformation between two reference frames. Lorentz (relativistic) transformation is an expansion of Galileo's (non- relativistic) transformation, according to the following picture:



[https://www.slideshare.net/HemBhattarai2/relativity-57673106]

In this picture motion occurs on the horizontal axes *x* and *x´*. Two observers are located at points *O* and *O´* of their respective reference frames *K* and *K´*. An event (red dot in the previous picture) takes place at some distance *x´* from point *O´*. The first reference frame *K* can be treated as stationary, while the second reference frame *K´* will be moving at the relative speed *v*. Initially the two reference frames may coincide, so that after some time *t* according to the observer on the first reference frame *K* (at point *O*), the distance between the two reference frames *K* and *K´* will be *OO´*=*vt*. If *x* is the distance (the coordinates) of the event from point *O*, and *x´* is its distance from point *O´*, then the Galilean transformation between the two reference frames is

$x = x' + vt$

$x' = x - vt$

The main difference between the Galilean transformation and Lorentz transformation in relativity, is that the times at which the same event takes place are different for the reference frames K and K'.

This happens because in relativity distances are measured with photons, whose speed c (the speed of light) is constant. The position x and x' of the event with respect to the two different reference frames K and K', respectively, will be

$$x = ct$$

$$x' = ct'$$

Thus the time t' according to the observer on the second reference frame K', will be different from the time t according to the observer on the first reference frame, as a consequence of the constant speed of light c.

Now a way to derive Lorentz transformation is to use the following linear combination,

$$x = \gamma_L(x' + vt')$$

$$x' = \gamma_L(x - vt)$$

where the Lorentz factor γ_L is added to the transformation as a correction factor.

Replacing in the previous formulas

$$x = ct$$

$$x' = ct'$$

we take

$$x = \gamma_L(x' + vt') \Rightarrow$$

$$ct = \gamma_L(ct' + vt') = \gamma_L(c + v)t',$$

$$x' = \gamma_L(x - vt) \Rightarrow$$

$$ct' = \gamma_L(ct - vt) = \gamma_L(c - v)t,$$

$$\frac{x}{x'} = \frac{ct}{ct'} = \frac{t}{t'} = \frac{\gamma_L(c+v)t'}{\gamma_L(c-v)t} = \frac{(c+v)t'}{(c-v)t} \Rightarrow$$

$$\frac{t^2}{t'^2} = \frac{c+v}{c-v} \Rightarrow$$

$$\frac{t}{t'} = \sqrt{\frac{c+v}{c-v}}$$

so that for the Lorentz factor γ_L we have

$$\gamma_L = \frac{ct'}{(c-v)t} = \frac{ct}{(c+v)t'} \Rightarrow$$

$$\gamma_L = \frac{c}{c+v}\sqrt{\frac{c+v}{c-v}} = \sqrt{\frac{c^2}{(c+v)^2}\frac{c+v}{c-v}} = \sqrt{\frac{c^2}{(c+v)(c-v)}} \Rightarrow$$

$$\gamma_L = \sqrt{\frac{c^2}{c^2-v^2}} = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}}$$

This is the value of Lorentz factor γ_L.

Intriguingly enough, the previous result relating the two coordinate times t and t' (the time as measured by the clocks of two observers on the different reference frames K and K'),

$$\frac{t}{t'} = \sqrt{\frac{c+v}{c-v}}$$

is identical to the one we earlier took for Doppler shift, which relates the frequencies (thus also the periods) of a light wave, according to two observers (one at rest, and another one moving at a relative speed v). That formula was given as

$$f_R = \sqrt{\frac{c \pm v}{c \mp v}} f_E = \sqrt{\frac{c+v}{c-v}} f_E$$

where here we have preferred a combination of signs implying that the observer (the receiver) approaches the source (the emitter), so that the frequency f_R he/she receives increases.

If instead of the frequencies f_R and f_E we use the corresponding periods T_R and T_E,

$$f_R = \frac{1}{T_R}, \qquad f_E = \frac{1}{T_E},$$

and make the appropriate replacements,

$$f_R \equiv f', \qquad f_E \equiv f$$

$$T_R \equiv T', \qquad T_E \equiv T$$

we can identify the periods T_R and T_E with the coordinate times t' and t, respectively,

$$\frac{f_R}{f_E} = \sqrt{\frac{c+v}{c-v}} = \frac{f'}{f} \Rightarrow$$

$$\frac{T_R}{T_E} = \frac{T'}{T} = \sqrt{\frac{c-v}{c+v}} \equiv \frac{t'}{t}$$

The times T_R and T_E (or T' and T respectively) refer to the period of photons, while the times t' and t refer to the clocks (measuring devices) of the observers. But if the times t' and t can be identified with the periods $T'$and T, respectively, then there must be an intricate relationship between the properties of the photons and the notion of time as we know it.

The deeper aspect of such a relationship, as we have already proposed, is that the photons with the altered frequency (as disturbances of spacetime) are produced by the motion of the observer in spacetime, in a cause- and- effect relationship.

But according to this description, the oscillating medium (spacetime) cannot be ignored. If we ignore the medium, we may end up with wrong results, or we may not be able to expand the result, in order to find more general formulas.

An example of such a generalization was given in the next to last section. There we derived a formula of the form

$$c^2T_2^2 = c^2T_1^2 - v^2(\Delta t)^2,$$

where the periods T_2 and T_1 (thus also the oscillations of spacetime measured in the form of photons) are related to the speed v of the observer, and according to his/her own clock Δt.

Here we can show the following thing. Replacing the periods T_2 and T_1, by T' and T respectively, and setting

$$\lambda = cT,$$

$$\lambda' = cT',$$

where λ and λ' represent the corresponding wavelengths of the oscillations (the photons), then we have that

$$cT'^2 = cT^2 - (v\Delta t)^2,$$

$$\lambda'^2 = \lambda^2 - (v\Delta t)^2$$

Synchronizing now the clock Δt of the moving observer with the initial period T (the period of photons he/she measures before he/she begins to move), we take for the periods,

$$\lambda'^2 = \lambda^2 - (v\Delta t)^2 \Rightarrow$$

$$(cT')^2 = (cT)^2 - (v\Delta t)^2 = (cT)^2 - (vT)^2, \qquad \Delta t \equiv T \Rightarrow$$

$$c^2T'^2 = (c^2 - v^2)T^2 \Rightarrow$$

$$T'^2 = \frac{c^2 - v^2}{c^2}T^2 \Rightarrow$$

$$T' = \sqrt{\frac{c^2 - v^2}{c^2}}T = \frac{1}{\gamma_L}T, \qquad \gamma_L = \sqrt{\frac{c^2}{c^2 - v^2}} = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}},$$

or, equivalently, for the wavelengths,

$$cT' = \frac{1}{\gamma_L}cT \Rightarrow$$

$$\lambda' = \frac{1}{\gamma_L}\lambda$$

Therefore by assuming that motion occurs in a medium, that this motion affects the medium by changing the period of its oscillations, and that we can measure this effect in the form of photons, we retrieve the previous relativistic expressions for length contraction (or time dilation), and for Lorentz factor γ_L, essentially from first principles, and directly from a quadratic equation of the general form

$$\lambda'^2 = \lambda^2 - (v\Delta t)^2$$

In the next section we will see that the same equation is valid in any occasion, even if we move at non- relativistic (much smaller than light) speeds.

Notes:

With respect to the general formula

$$c^2T_1^2 = c^2T_2^2 + v^2(\Delta t)^2,$$

or equivalently

$$c^2T^2 = c^2T'^2 + v^2(\Delta t)^2,$$

where

$$T_1 \equiv T, \qquad T_2 \equiv T', \qquad \Delta t = T + T'$$

solving for the periods T, or T', we take

$$c^2T^2 = c^2T'^2 + v^2(\Delta t)^2 \Rightarrow$$

$$v^2(\Delta t)^2 = c^2T^2 - c^2T'^2 \Rightarrow$$

$$v^2(T + T')^2 = c^2T^2 - c^2T'^2 = c^2(T^2 - T'^2) = c^2(T + T')(T - T') \Rightarrow$$

$$v^2(T + T') = c^2(T - T') \Rightarrow$$

$$v^2T + v^2T' = c^2T - c^2T' \Rightarrow$$

$$(c^2 - v^2)T = (c^2 + v^2)T' \Rightarrow$$

$$T' = \frac{c^2 - v^2}{c^2 + v^2} T$$

If instead of

$$\Delta t = T + T'$$

we set

$$(\Delta t)^2 = T^2 + T'^2$$

assuming thus a position vector of the form

$$c^2(\Delta t)^2 = c^2T^2 + c^2T'^2$$

then we alternatively take

$$c^2T^2 = c^2T'^2 + v^2(\Delta t)^2 \Rightarrow$$

$$v^2(T^2 + T'^2) = c^2T^2 - c^2T'^2 \Rightarrow$$

$$v^2T^2 + v^2T'^2 = c^2T^2 - c^2T'^2 \Rightarrow$$

$$c^2T^2 - v^2T^2 = c^2T'^2 + v^2T'^2 \Rightarrow$$

$$(c^2 - v^2)T^2 = (c^2 + v^2)T'^2 \Rightarrow$$

$$T'^2 = \frac{c^2 - v^2}{c^2 + v^2} T^2 \Rightarrow$$

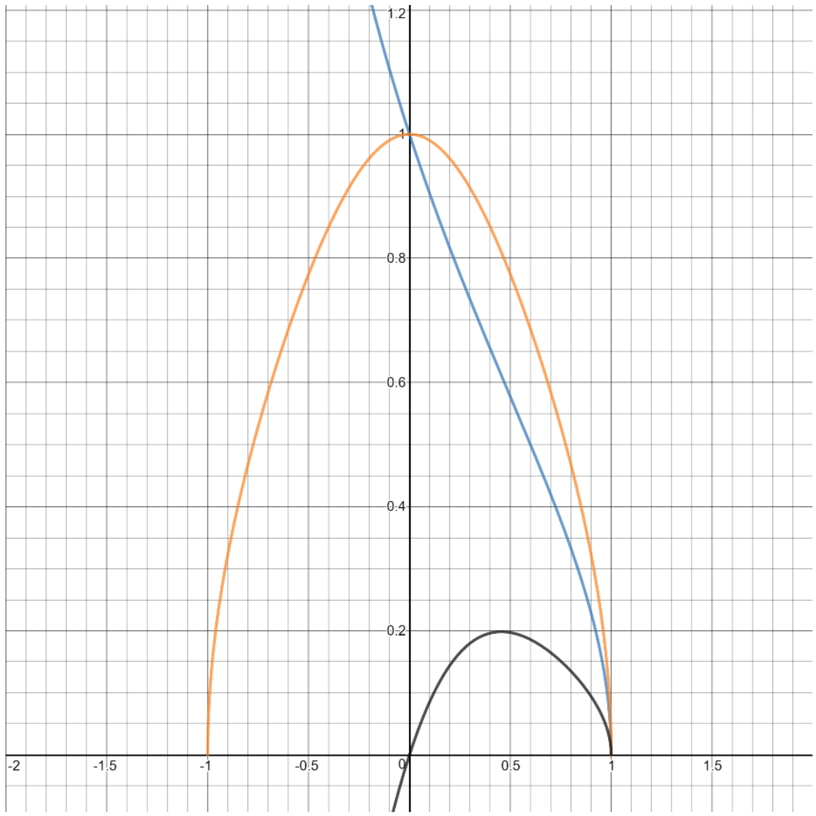
$$T' = \sqrt{\frac{c^2 - v^2}{c^2 + v^2}} T$$

This expression, compared to the expression for Doppler shift,

$$T' = \sqrt{\frac{c - v}{c + v}} T$$

has the advantage of being more symmetrical, as the speed *v* increases.

This can be seen in the following graph:



Blue: $T' = \sqrt{\frac{c-v}{c+v}}T \approx \sqrt{\frac{1-v}{1+v}}T$

Orange: $T' = \sqrt{\frac{c^2-v^2}{c^2+v^2}}T \approx \sqrt{\frac{1-v^2}{1+v^2}}T$

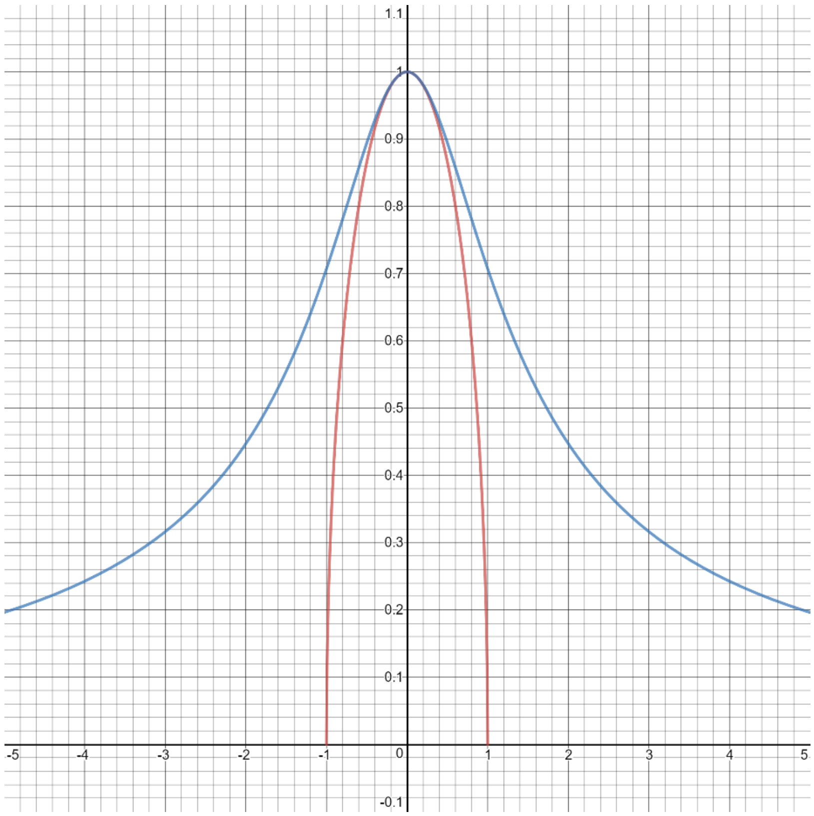
Black: $\sqrt{\frac{c^2-v^2}{c^2+v^2}}T - \sqrt{\frac{c-v}{c+v}}T \approx \sqrt{\frac{1-v^2}{1+v^2}}T - \sqrt{\frac{1-v}{1+v}}T,$

$c = 1$

The graph compares the function for the relativistic Doppler shift (blue line), to the function we previously mentioned (orange line). The black line is the difference between these two functions. As the graph suggests, this difference is bigger at about *v=0.5c*.

Such a difference can be measurable, so that it can be tested if the second formula corresponds better to reality.

More importantly, however, we may plot the following graph:



Red: $T' = \sqrt{1 - \frac{v^2}{c^2}}T \approx \sqrt{1 - v^2}T$

Blue: $T' = \sqrt{\frac{1}{1+\frac{v^2}{c^2}}}T \approx \sqrt{\frac{1}{1+v^2}}T,$

$c = 1$

This graph compares the function (blue line)

$c^2T'^2 = c^2T^2 - v^2(\Delta t)^2 \Rightarrow$

$$T' = \sqrt{\frac{1}{1 + \frac{v^2}{c^2}}}T = \gamma_L^+ T, \qquad \Delta t \equiv T',$$

to the common relativistic expression (red line)

$$T' = \sqrt{1 - \frac{v^2}{c^2}}T = \frac{1}{\gamma_L^-}T, \qquad v \ll c$$

where

$$\gamma_L^+ = \sqrt{\frac{1}{1+\frac{v^2}{c^2}}}, \qquad \gamma_L^- = \sqrt{\frac{1}{1+\frac{v^2}{c^2}}}$$

The fundamental difference between the two functions is that while in the case of the classical relativistic expression the period T' (thus also the corresponding time t') will become zero for the moving observer if his/her speed v becomes equal to the speed of light c, in the case of the expression with the positive Lorentz factor $\gamma_L{}^+$ the same period T' will become zero for the moving observer if his/her speed v becomes infinite.

Such an aspect is shown in the graph, as the red line becomes zero at $v=c$ ($c=1$ in the graph), while the blue line becomes zero if $v \to \infty$. This is an illustrative way to show, what we have already mathematically derived, the possibility of faster than light travel.

The ship in the sea

Here we will show how the relativistic expression for length contraction, or time dilation, can be derived in a physical context, thus with respect to some medium, which, while in the general case we can identify with spacetime, in the special case of this example will be the sea itself.

Motion in the sea is in fact an oscillatory motion, as the ship follows the oscillations of the waves. Usually we ignore the curvature (amplitude) of the sea waves, thus we also ignore the effect of time dilation related to the period of the waves. Instead we estimate the time arrival of the ship as if the ship were travelling on a straight line (in fact this is true only if we travel at a speed equal to the speed of the waves). However if, for example, the sea is rough, so that the amplitude of the waves is large, the effect of time dilation can be significant.

In order to describe the problem, this is a related graph:

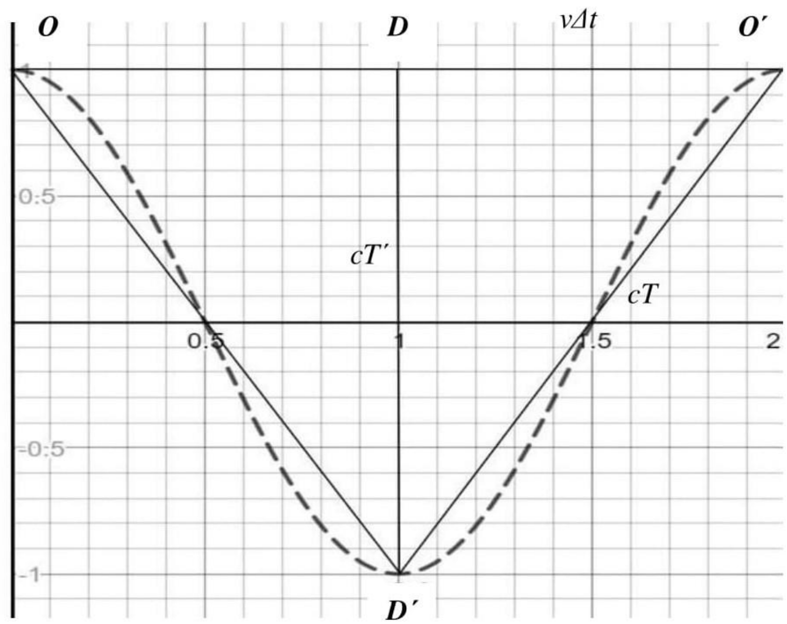


Illustration of the ship- in- the- sea thought experiment

According to the previous picture, the ship is initially located at point O, on the crest of the wave. An observer on board the ship measures the period T of the wave at rest (when the ship has no speed of its own). If the speed of the wave is c, then at a time equal to $T/2$ the wave moves a linear distance $OD=c\Delta T/2$ (half a wavelength). As the ship begins to move at a speed v, the period of the

wave changes to T', according to an observer on board the ship. Presumably, if there was no wave, after a time interval $\Delta t/2$, as measured by the clock of the observer, equal to the period $\Delta T/2$ of the wave at rest, the ship would have traveled a horizontal distance $OD=v\Delta T/2$. But because the wave is oscillating, the ship is delayed by the vertical distance $DD=cT'/2$ of the oscillation.

Here the distances are measured with respect to the period of the waves,

$$\frac{x}{2} = OD = v\frac{\Delta t}{2} = v\frac{T}{2}, \qquad \Delta t \equiv T,$$

$$\frac{\lambda}{2} \approx OD' = O'D' = c\frac{T}{2},$$

$$\frac{\lambda'}{2} \approx DD' = c\frac{T'}{2}$$

Thus the time dilation is given as follows,

$$(OD')^2 = (DD')^2 + (OD)^2 \Rightarrow$$

$$\left(c\frac{T}{2}\right)^2 = \left(c\frac{T'}{2}\right)^2 + \left(v\frac{\Delta T}{2}\right)^2 \Rightarrow$$

$$c^2T^2 = c^2T'^2 + v^2T^2 \Rightarrow$$

$$c^2T'^2 = c^2T^2 - v^2T^2 = (c^2 - v^2)T^2 \Rightarrow$$

$$T'^2 = \frac{c^2 - v^2}{c^2}T^2 \Rightarrow$$

$$T' = \sqrt{\frac{c^2 - v^2}{c^2}}T = \sqrt{1 - \frac{v^2}{c^2}}T \Rightarrow$$

$$T' = \frac{1}{\gamma_L}T, \qquad \gamma_L = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}$$

This result is identical to that in Einstein's train thought experiment, if we replace the periods T and T' by the coordinate times t and t', respectively, of two observers.

However there is one basic exception. The times T and T' refer to the periods of the wave (the medium), not to the period of photons (the time it takes photons to travel the distances).

Thus the effect of time dilation (or length contraction) can be directly attributed to the motion of the medium (even at conventional speeds). Therefore the relativistic formula, which we have already seen in previous sections, finds here a physical meaning.

This meaning is in accordance with the common experience we have when moving in the sea, that the frequency of the waves which come to us increases (or that the period, thus also their wavelength, decreases).

Incidentally, in the previous description, it was assumed that the speed of the sea waves was c. But if we replace the sea with spacetime, then, supposedly, the speed c will be the speed of light.

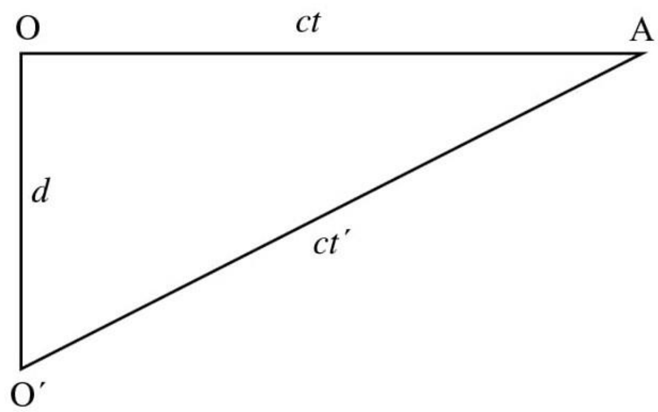
That spacetime may produce waves is not absurd. If there is energy stored in spacetime, then this energy will be expressed with some form of oscillations, therefore there will be a wave associated with spacetime.

Sea waves are in fact gravitational waves (so that water helps us perceive these waves with our own eyes). Thus spacetime can be seen as a gravitational wave, composed by gravitons instead of molecules of water.

That the oscillation of such a medium also produces photons- by which in fact we can measure the spatial and temporal properties of the oscillating medium- also poses the question which is the relationship between gravitons and photons. An attempt to answer this question will be made later on, after we introduce the notion of the brachistochrone.

Finally, the intimate relationship between the periods T and T' of the medium (spacetime) and the coordinate times t and t', of two observers at rest and in motion, respectively, also exposes the deeper connection between physical reality and our own experience of the same reality. Such a connection will be explored at the end of this document.

Notes:



A way to measure a gravitational wave is using an interferometer, which oscillates as the gravitational wave passes by. If the interferometer is placed at point O (previous picture), then, if d is the amplitude of the gravitational wave, the instrument will be displaced at point O'. The displacement, thus also the amplitude of the gravitational wave, is measured in relation to the distances (spacetime intervals) OA and $O'A$. These distances are associated with photons, and with the times t and t', respectively, it takes the photons to travel the distances. Thus the amplitude of the gravitational wave will be

$$(OO')^2 = (O'A)^2 - (OA)^2 \Rightarrow$$
$$d^2 = c^2t'^2 - c^2t^2$$

If we also assume that the gravitational wave oscillates at the speed of light c, then, if T is the period of the gravitational wave, it will be

$$d = cT \Rightarrow$$
$$c^2T^2 = c^2t'^2 - c^2t^2 \Rightarrow$$
$$t'^2 - t^2 = T^2$$

In that sense, the time as measured by the observer who uses the interferometer can be defined with respect to the period of the gravitational wave.

Similarly, the two times t and t' can be related to two different observers, located at points (reference frames) O and O', respectively. If d is the amplitude difference of the gravitational wave

between those two points, then the previous equation will give the time difference on the clocks of the two observers.

If we now attribute the gravitational wave to the motion of an object (which produces the gravitational wave by its motion), and relate the two times t and t' to the period of photons with which the location of the object is measured, then we go back to the case of the ship in the sea, as earlier described.

The simple harmonic oscillator

The motion of a ship in the sea can be compared to that of the simple harmonic oscillator. In fact the aspect that the natural (initial) period of the wave T is reduced to T' implies that as the object (the ship) moves, it pumps energy from the wave. If the wave is described by a simple harmonic oscillator, the equation of motion is written as

$$ma + ky = 0 \Rightarrow$$

$$a + \frac{k}{m}y = 0 \Rightarrow$$

$$a + \omega_0^2 y = 0, \qquad \omega_0^2 = \frac{k}{m}$$

and has a solution of the form

$$y = y_0 \cos \omega_0 t$$

$$v = -\omega_0 y_0 \sin \omega_0 t = -v_0 \sin \omega_0 t\,, \qquad v_0 = \omega_0 y_0$$

$$a = -\omega_0^2 y_0 \cos \omega_0 t = -a_0 \cos \omega_0 t\,, \qquad a_0 = \omega_0^2 y_0 = \omega_0 v_0$$

where y_0 is the (maximum) amplitude at $t=0$, and ω_0 is the natural frequency of the oscillator.

The associated energies, the elastic energy E_{el}, the kinetic energy E_k, and the mechanical (total) energy E_m (which we may also call E_0) of the system are given as follows,

$$E_{el} = \frac{1}{2}ky^2 = \frac{1}{2}m\omega_0^2 y_0^2 \cos^2 \omega_0 t\,, \qquad k = m\omega_0^2$$

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega_0^2 y_0^2 \sin^2 \omega_0 t = \frac{1}{2}mv_0^2 \sin^2 \omega_0 t\,, \qquad v_0 = \omega_0 y_0$$

$$E_{el} = E_{el} + E_k = \frac{1}{2}m\omega_0^2 y_0^2 \cos^2 \omega_0 t + \frac{1}{2}m\omega_0^2 y_0^2 \sin^2 \omega_0 t \Rightarrow$$

$$E_m = \frac{1}{2}m\omega_0^2 y_0^2(\cos^2 \omega_0 t + \sin^2 \omega_0 t) = \frac{1}{2}m\omega_0^2 y_0^2 = \frac{1}{2}mv_0^2 \equiv E_0,$$

$$v_0 = \omega_0 y_0, \qquad \cos^2 \omega_0 t + \sin^2 \omega_0 t = 1$$

Later on we will include damping and a driving force into the simple harmonic oscillator.

In order to see how we can retrieve the relativistic formula for time dilation, we rewrite the energy equation of the simple harmonic oscillator in the following equivalent way,

$$E_0 = E_{el} + E_k \Rightarrow$$

$$\frac{1}{2}ky_0^2 = \frac{1}{2}ky^2 + \frac{1}{2}mv^2 \Rightarrow$$

$$\frac{1}{2}m\omega_0^2 y_0^2 = \frac{1}{2}m\omega_0^2 y^2 + \frac{1}{2}mv^2, \qquad k = m\omega_0^2$$

Simplifying this expression, we have,

$$\frac{1}{2}m\omega_0^2 y_0^2 = \frac{1}{2}m\omega_0^2 y^2 + \frac{1}{2}mv^2 \Rightarrow$$

$$\omega_0^2 y_0^2 = \omega_0^2 y^2 + v^2 \Rightarrow$$

$$y_0^2 = y^2 + \frac{v^2}{\omega_0^2} \Rightarrow$$

$$\frac{y_0^2}{y_0^2} = \frac{y^2}{y_0^2} + \frac{v^2}{\omega_0^2 y_0^2} \Rightarrow$$

$$1 = \frac{y^2}{y_0^2} + \frac{v^2}{\omega_0^2 y_0^2} \Rightarrow$$

$$\frac{y^2}{y_0^2} = 1 - \frac{v^2}{\omega_0^2 y_0^2}$$

Here we will use the notion of the reference circle as the geometric approximation of an oscillation. If y_0 is the amplitude of the oscillation, and λ_0 is the wavelength, then at a time equal to the period T_0 of the oscillation, a point on the wave moves on a circle whose radius is the amplitude y_0, and whose perimeter is the wavelength λ_0, so that $\lambda_0 = 2\pi y_0$. Therefore, going back to the previous equation,

$$\frac{y^2}{y_0^2} = 1 - \frac{v^2}{\omega_0^2 y_0^2} \Rightarrow$$

$$\frac{y^2}{y_0^2} = 1 - \frac{v^2}{\omega_0^2 \frac{\lambda_0^2}{4\pi^2}}, \qquad y_0 = \frac{1}{2\pi}\lambda_0 \Rightarrow$$

$$\frac{y^2}{y_0^2} = 1 - \frac{v^2}{\frac{4\pi^2}{T_0^2}\frac{c^2 T_0^2}{4\pi^2}}, \qquad \omega_0 = \frac{2\pi}{T_0} \Rightarrow$$

$$\frac{y^2}{y_0^2} = 1 - \frac{v^2}{c^2} \Rightarrow$$

$$\frac{y}{y_0} = \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma_L}, \qquad \gamma_L = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}}$$

where γ_L is Lorentz factor.

The last expression can also be written with respect to the times T_0 and T,

$$\frac{y}{y_0} \approx \frac{2\pi\lambda}{2\pi\lambda_0} = \frac{\lambda}{\lambda_0} = \frac{cT}{cT_0} = \frac{T}{T_0} \Rightarrow$$

$$\frac{T}{T_0} = \sqrt{1 - \frac{v^2}{c^2}}$$

This way we have retrieved the relativistic expression relating the two times, the only difference being that here the two times are T_0 and T, instead of T and T', respectively, according to the notation we have used earlier.

Later on we shall see that the reduction of the period of the wave is due to damping.

Notes:

The equations of the simple harmonic oscillator normally refer to a mass *m* hanging from a spring, where *k* is the spring constant. But if the ‘spring’ is the sea, and the mass ‘hanging from the spring’ is the ship oscillating in the wave, then we can describe the motion of the ship in the sea with the equation of the simple harmonic oscillator.

Furthermore these equations, as used here, refer to the vertical displacement of the ship,

$$y = y_0 \cos \omega_0 t,$$

$$\Delta y = y_0 - y = y_0 - y_0 \cos \omega_0 t = y_0 (1 - \cos \omega_0 t)$$

But since the ship follows the motion of the wave, it is displaced on the vertical axis Δy as much as it is displaced on the horizontal axis Δx. Therefore the displacement on the horizontal axis can be found by replacing y with x in the previous equations.

The rocket in space

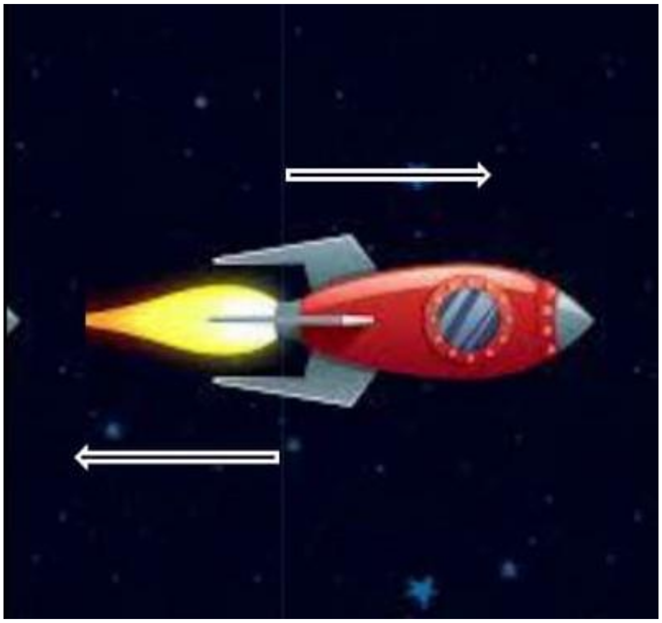


Illustration of the ‘rocket in space:’ The rocket pushes ‘empty’ space backwards with a force $F=-bv$, while ‘empty’ space pushes the rocket forward with an equal but opposite force $F=bv$.

Once it was believed that the motion of a rocket in empty space would have been impossible, because the rocket would have had nothing to push (e.g. the air). But this was a misinterpretation of the true phenomenon, because, even in the simplest case of a car, it is not the exhaustion gases which push the air, so that the vehicle moves in the opposite direction, but it is the loss of mass of the vehicle, which changes the vehicle’s speed (thus also the vehicle’s momentum).

This aspect rises from the conservation of the momentum of the object. For example, if m_1 and v_1 is the initial mass and speed of the vehicle, and m_2 and v_2 is its final mass and speed, respectively, then the conservation of the momentum p gives

$$\Delta p = 0 \Rightarrow$$

$$p_2 - p_1 = 0 \Rightarrow$$

$$m_2 v_2 - m_1 v_1 = 0 \Rightarrow$$

$$m_2 v_2 = m_1 v_1 \Rightarrow$$

$$v_2 = \frac{m_1}{m_2} v_1$$

If the vehicle has lost mass in the form of burnt fuel, then it will be

$$m_2 < m_1 \Rightarrow$$
$$v_2 > v_1$$

so that the moving object increases its speed simply because it burns fuel (thus loses mass).

Still, if we use the notion of the force, the change of the moving object's momentum will be equivalent to a force (a thrust) produced by the object. This is because the force is the derivative of the momentum with respect to time, according to Newton's second law of motion. Also, according to Newton's third law of action and reaction, if a rocket moving in empty space exerts a thrust on empty space, then empty space will push the rocket with an equal but opposite force. However, this raises the question of what nature is such a force, and, more importantly, how come empty space produces forces, if space is empty.

So let's take the rocket- in- space example. If the rocket has a mass *m*, and moves at a speed *v*, using Newton's second law, the equation of motion of the rocket will be

$$\sum F = ma \Rightarrow$$
$$-bv = ma \Rightarrow$$
$$ma + bv = 0$$

That the force the rocket produces is $F=-bv$, can be seen if we take the full expression (if the mass also changes) of Newton's second law

$$\sum F = \frac{dp}{dt} = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} = ma + bv, \qquad a = \frac{dv}{dt}, \qquad b = \frac{dm}{dt}$$

where $p=mv$ is the rocket's momentum.

If the momentum is conserved, then

$$\sum F = \frac{dp}{dt} = 0 \Rightarrow$$
$$ma + bv = 0$$

which gives us back the equation of motion earlier mentioned.

Here we should mention that the force *F=-bv* has the aspect of a resistance force, which opposes the motion of the rocket, if the medium is not empty. Such a force is usually considered to be proportional to the speed *v* of the moving object, where the constant of proportionality is *b*.

However, at the same time, this force can also be seen as the thrust which the rocket produces, and imposes on the medium. In such a sense this force will be a damping force, proportional again to the speed *v* of the object, where here the constant of proportionality *b* will be the damping constant.

Thus the constant *b* is related to a property of the medium (its resistance), as much as to a property of the object which moves in the medium (damping), since it represents the rate with which the object burns mass (in the form of fuel),

$$b = \frac{dm}{dt}$$

If we now rewrite the previous equation of motion in the following equivalent form,

$$ma + bv = 0$$

$$a + \frac{b}{m}v = 0 \Rightarrow$$

$$a + \gamma v = 0,$$

$$\gamma = \frac{b}{m} = \frac{dm}{dt}\frac{1}{m}$$

we can identify the parameter γ with the damping factor, which we will often encounter hereafter.

The point here is that spacetime is not empty, but that it has a certain amount of energy, thus also a certain amount of mass associated to that energy. In the next section we will see that this mass is part of the moving object's mass, in the form of its fuel content. Thus when the object burns its fuel in order to move, it consumes a part of the mass (or equivalently energy) which is stored in spacetime.

Returning now to the equation of motion,

$$ma + bv = 0, \qquad a + \gamma v = 0, \qquad \gamma = \frac{b}{m}, \qquad a = \frac{dv}{dt}$$

we may apply the following solution,

$$y(t) = y_0 e^{-\gamma t}$$

$$v(t) = -\gamma y_0 e^{-\gamma t} = -\gamma y(t) = -v_0 e^{-\gamma t}, \qquad v_0 = \gamma y_0$$

$$a(t) = \gamma^2 y_0 e^{-\gamma t} = \gamma^2 y(t) = -\gamma v(t) = a_0 e^{-\gamma t}, \qquad a_0 = \gamma^2 y_0 = \gamma v_0$$

Substituting the solution into the equation of motion, we take,

$$ma + bv = 0 \Rightarrow$$

$$m\gamma^2 y_0 e^{-\gamma t} - b\gamma y_0 e^{-\gamma t} = 0 \Rightarrow$$

$$m\gamma^2 - b\gamma = 0 \Rightarrow$$

$$m\gamma^2 = b\gamma \Rightarrow$$

$$\gamma = \frac{b}{m}$$

so that the proposed solution is valid as long as the damping factor γ is given by the previous expression.

Now with respect to the kinetic energy, we have by definition

$$E_k(t) = \int mady = \int m\frac{dv}{dt}dy = \int mdv\frac{dy}{dt} = \int mvdv = \frac{1}{2}mv^2(t)$$

so that in our case

$$E_k(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}mv_0^2 e^{-2\gamma t} = \frac{1}{2}m\gamma^2 y_0^2 e^{-2\gamma t} = \frac{1}{2}m\gamma^2 y^2(t),$$

$$v_0 = \gamma y_0, \qquad y(t) = y_0 e^{-\gamma t}$$

Setting $t=0$ as initial condition, we have

$$y(t) = y_0 e^{-\gamma t} \Rightarrow y(0) = y_0$$

$$v(t) = -\gamma y_0 e^{-\gamma t} \Rightarrow v(0) = -\gamma y_0 = -v_0, \qquad v_0 = \gamma y_0$$
$$a(t) = \gamma^2 y_0 e^{-\gamma t} \Rightarrow a(0) = \gamma^2 y_0 = \gamma v_0 = a_0, \qquad a_0 = \gamma^2 y_0 = \gamma v_0$$

so that the initial (total) energy will be

$$E_0 \equiv E(t=0) = \frac{1}{2} m\gamma^2 y_0^2 = \frac{1}{2} m v_0^2$$

We may notice that the term

$$E_d(t) = \frac{1}{2} m\gamma^2 y^2(t) = \frac{1}{2} m\gamma^2 y_0^2 e^{-2\gamma t}$$

can be identified with the measure of the energy related to the damping, thus the damping energy.

More accurately, the damping energy E_d is given by the following integral,

$$E_d(t) = \int bvdy = \int \frac{dm}{dt} vdy = \int dmv\frac{dy}{dt} = \int dmv^2\,, \qquad b = \frac{dm}{dt} = \gamma m, \qquad v = \frac{dy}{dt}$$

But if in the last integral we treat the mass m of the rocket, as well as the damping factor γ, as constants, then we may write

$$E_d(t) = \int bvdy = \int \gamma mvdy = \int \gamma m(-\gamma y)dy = -\gamma^2 m \int ydy,$$
$$b = \gamma m, \qquad v(t) = -\gamma y(t) \Rightarrow$$
$$E_d(t) = -\frac{1}{2} m\gamma^2 y^2(t)$$

so that we take an energy equation of the form

$$\frac{1}{2} mv^2(t) + \frac{1}{2} m\gamma^2 y^2(t) = E_0,$$
$$E_0 = \frac{1}{2} mv_0^2 = \frac{1}{2} m\gamma^2 y_0^2$$

The equivalence between the two forms of energy, the kinetic energy E_k, and the damping energy E_d, can also be seen by the rate of change of the energies,

$$\frac{dE_k}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{1}{2}m\frac{d}{dt}[v^2(t)] = \frac{1}{2}m[2v(t)a(t)] = mva$$

where

$$mva = mv(-\gamma v) = -m\gamma v^2 = -bv^2, \qquad b = \gamma m$$

If the total energy of the system is conserved then one form of energy will be transformed into the other form, so that it will be

$$\frac{dE_k}{dt} = -\frac{dE_d}{dt} \Rightarrow$$

$$\frac{dE_k}{dt} = -bv^2$$

If we multiply and divide the term on the right hand side of the last equation by the kinetic energy, we take

$$\frac{dE_k(t)}{dt} = -bv^2(t) = -bv^2(t)\frac{E_k(t)}{\frac{1}{2}mv^2(t)} = -2\frac{b}{m}E_k(t) = -2\gamma E_k(t),$$

$$\gamma = \frac{b}{m}$$

The same result can be taken more directly if we simply multiply both terms in the equation of motion by *v*,

$$ma = -bv \Rightarrow$$

$$mav = -bv^2 \Rightarrow$$

$$\frac{dE_k(t)}{dt} = -bv^2(t) = -2\gamma E_k(t),$$

$$E_k(t) = \frac{1}{2}mv^2(t)$$

If we integrate the last differential equation, we will take back the expression for the kinetic energy in exponential form,

$$\frac{dE_k(t)}{dt} = -bv^2(t) = -2\gamma E_k(t) \Rightarrow$$

$$\frac{dE_k(t)}{E_k(t)} = -2\gamma dt \Rightarrow$$

$$\int \frac{dE_k(t)}{E_k(t)} = -2\gamma \int dt \Rightarrow$$

$$\ln E_k(t) = -2\gamma t + C \Rightarrow$$

$$\ln E_k(t) = -2\gamma t + \ln E_0\,, \qquad C = \ln E_0 \Rightarrow$$

$$\ln E_k(t) - \ln E_0 = \ln \frac{E_k(t)}{E_0} = -2\gamma t \Rightarrow$$

$$\frac{E_k(t)}{E_0} = e^{-2\gamma t} \Rightarrow$$

$$E_k(t) = E_0 e^{-2\gamma t},$$

$$E_0 = \frac{1}{2} m v_0^2 = \frac{1}{2} m \gamma^2 y_0^2$$

The same equation may be written without an index, by simply writing

$$E(t) = E_0 e^{-2\gamma t}$$

In that form, because the energy decays with time due to the negative exponential, we may associate it with the (damping) energy which spacetime loses, so that the kinetic energy of the rocket moving in spacetime will more appropriately be given by the difference

$$\varDelta E(t) = E_0 - E(t) = E_0 - E_0 e^{-2\gamma t} = E_0(1 - e^{-2\gamma t})$$

We will meet again such a solution and energy equation, when the harmonic force *ky* will be included into the equation of motion, which is the case of the damped harmonic oscillator. There, when critical damping is applied, the equivalence between the forms of energy will become more obvious.

Notes:

We have already mentioned the aspect that spacetime contains an amount of energy. We shall see in the next section that such energy can be represented by a mass μ_0 of the wave of spacetime, different than the inertial mass m_0 of the object moving in spacetime.

Here we will make the comparison between the kinetic energy $\Delta E(t)$ which was previously mentioned, and the energy in relativity. The relativistic energy is given by the following expression,

$$E_{Ein} = mc^2 = \gamma_L m_0 c^2 = \gamma_L E_0,$$
$$m = \gamma_L m_0$$

where m is the relativistic mass (which presumably increases with the speed v), and m_0 is the 'rest mass.'

Accordingly, the energy $E_{Ein}=mc^2$ (where the index '*Ein*' stands for '*Einstein*') is the total energy, while E_0 is the 'rest energy.' In that sense, the quantity

$$\Delta E_{Ein} = mc^2 - m_0c^2 = \gamma_L m_0 c^2 - m_0 c^2 = m_0 c^2(\gamma_L - 1) = E_0(\gamma_L - 1)$$

will be the kinetic energy.

This kinetic energy can be reduced to the classical kinetic energy $E_k=1/2mv^2$, at the limit of small speeds, in the following way. Taking the linear approximation of the Lorentz factor, if the speed v of the object is small compared to the speed of light c, $v<<c$, $v\approx 0$, we have

$$\gamma_L = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}} = \left(\frac{1}{1-\frac{v^2}{c^2}}\right)^{\frac{1}{2}} = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}, \qquad v \ll c$$

so that

$$\Delta E_{Ein} = E_0(\gamma_L - 1) = m_0c^2(\gamma_L - 1) \approx m_0c^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) = m_0c^2\left(\frac{1}{2}\frac{v^2}{c^2}\right), \qquad v \ll c \Rightarrow$$

$$\Delta E_{Ein} = \frac{1}{2}m_0v^2 \equiv E_k$$

The problem however with Einstein's energy, E_{Ein}, is that it goes to infinity if the speed v of the moving object reaches the speed of light c,

$$\gamma_L = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}} \approx \sqrt{\frac{1}{1-1}} = \sqrt{\frac{1}{0}} \to \infty, \qquad v \approx c$$

Now we will see how this problem may be treated.

In order to compare the relativistic expression for the energy, to the energy in the exponential form
$E(t) = E_0 e^{-2\gamma t}$

we set
$\gamma t = \frac{1}{4}n = \frac{1}{4}\frac{v}{c}$

where the ¼ factor, which is not significant, was added in order to have a factor of ½ appear in the exponential.

Thus we take
$E(t) = E_0 e^{-2\gamma t} \Rightarrow$

$E(n) = E_0 e^{-\frac{1}{2}n} \Rightarrow$

$E(v) = E_0 e^{-\frac{1v}{2c}}$

We can rewrite the relativistic factor as

$$\gamma_L = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}} = \sqrt{\frac{1}{1-\beta_L^2}}, \qquad \beta_L = \frac{v}{c} = \sqrt{1-\frac{1}{\gamma_L^2}}, \qquad 1-\beta_L^2 = \frac{1}{\gamma_L^2}$$

so that

$E(v) = E_0 e^{-\frac{1v}{2c}} = E_0 e^{-\frac{1}{2}\beta_L}$

We should not confuse the relativistic factor γ_L (Lorentz factor) with the damping factor $\gamma=b/m$, or β_L with $b=dm/dt$.

If β_L is sufficiently small, so that $\beta_L=v/c\approx 0$, $v<<c$, then the linear approximation of the exponential in the previous expression will be

$$E(v) = E_0 e^{-\frac{1}{2}\beta_L} = E_0 e^{-\frac{1}{2}\frac{v}{c}} \approx E_0\left(1-\frac{1}{2}\frac{v}{c}\right) \approx E_0\left(1-\frac{v}{c}\right)^{\frac{1}{2}} \approx E_0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}},$$

$$\beta_L \approx 0, \qquad v \ll c$$

The last approximation is based on the fact that if v is sufficiently small, then setting $x=v/c$, we have

$$(1-x)^2 \approx 1-x^2, \qquad x \approx 0 \Rightarrow$$

$$1-x \approx \sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}} = \left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} = (1-\beta_L^2)^{\frac{1}{2}} = \frac{1}{\gamma_L},$$

$$x = \frac{v}{c} \equiv \beta_L \approx 0, \qquad v \ll c$$

Thus we take that

$$E(v) \approx E_0\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}} \Rightarrow$$

$$E(v) \approx E_0\frac{1}{\gamma_L}$$

The last expression is the same as the relativistic expression

$$E_{Ein} = \gamma_L E_0$$

taking into account that in our expression E_0 refers to the total energy, while E_0 in the relativistic expression refers to the 'rest energy,' thus the minimum energy.

Therefore we take back the relativistic expression at the limit of small speeds.

On the other hand, as the speed v of the object approaches the speed of light c, we have that

$$E(v) = E_0 e^{-\frac{1}{2}\frac{v}{c}} = E_0 e^{-\frac{1}{2}\beta_L}$$

$$v = c, \qquad \gamma_L = \infty, \qquad \beta_L = 1 \Rightarrow$$

$$E(v) = E_0 e^{-\frac{1}{2}} = \frac{1}{2e} E_0,$$

$$\Delta E(v) = E_0 - E(v) = \left(1 - \frac{1}{2e}\right) E_0$$

Thus at the speed of light, $v \approx c$, the energy is reduced just by a factor of $1/e$ (or $1/2e$).

Additionally, at speeds greater than light, the exponential e^{-x} behaves like $1/x^2$ (or $1/2x^2$),

$$e^{-x} \approx \frac{1}{x^2}, \qquad x \gg 1,$$

$$e^{-n} \approx \frac{1}{n^2}, \qquad n \gg 1, \qquad n = \frac{v}{c} = v, \qquad c = 1$$

so that for the energy we take

$$E(n) = E_0 e^{-\frac{1}{2}n}, \qquad n = \frac{v}{c}$$

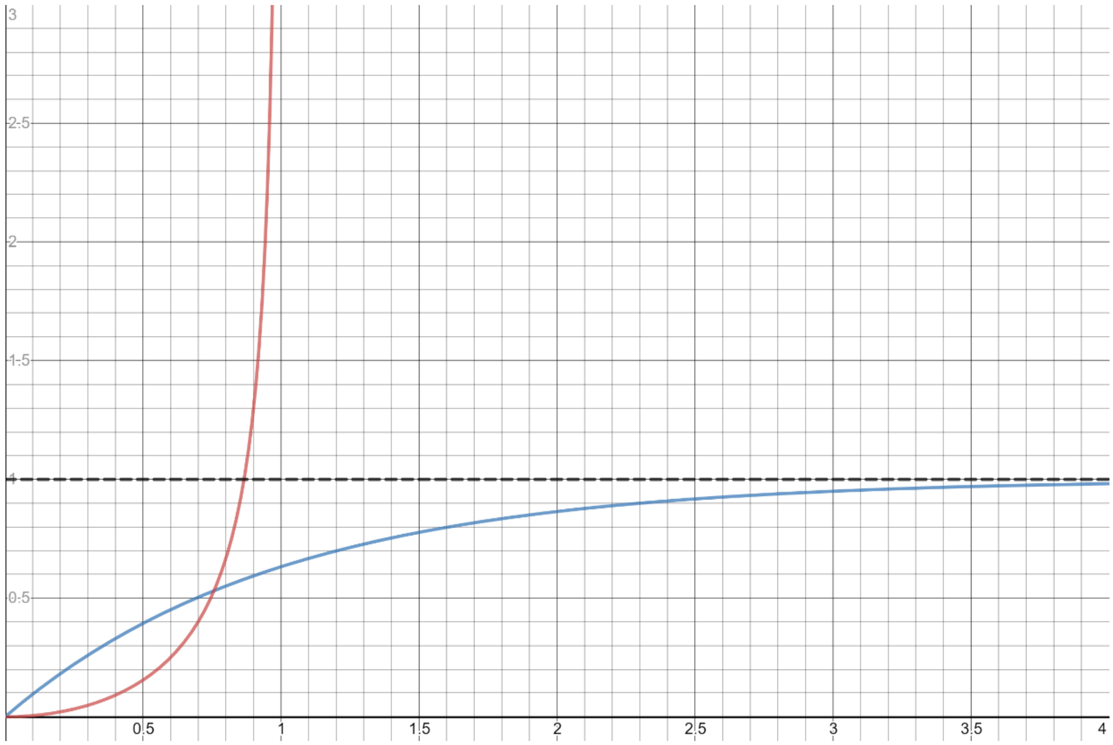
$$v \gg c, \qquad n \to \infty \Rightarrow$$

$$E(n) = E_0 e^{-\frac{1}{2}n} \approx \frac{1}{n^2} E_0,$$

$$\Delta E(n) \approx \left(1 - \frac{1}{n^2}\right) E_0$$

Thus the energy goes to zero only if the speed of the moving object goes to infinity $v \to \infty$.

This is a graph comparing the difference between the two energies:



Red: $\Delta E_{Ein}(v) = E_0(\gamma_L - 1) = E_0\left(\sqrt{\frac{1}{1-\frac{v^2}{c^2}}} - 1\right) \approx \sqrt{\frac{1}{1-v^2}} - 1$

Blue: $\Delta E(v) = E_0\left(1 - e^{-\frac{v}{c}}\right) \approx (1 - e^{-v})$

$E_0 = 1, \qquad c = 1$

In the previous graph we see how the relativistic expression for the kinetic energy (red line) goes to infinity, as the speed v approaches the speed of light c ($c=1$ in the graph), while the kinetic energy in the exponential form approaches asymptotically the maximum value E_0 ($E_0=1$ in the graph), as the speed v approaches infinity.

This is an example of how the problem of the infinite energy in relativity can be treated, even if the speed v of the moving object exceeds the speed of light c.

In summary, by relating the damping factor γ to Lorentz factor γ_L, we came up with an expression for the energy of the moving object which can be defined even if the object moves faster than light.

This was the original purpose of this document. The relationship between the properties of a material object, and the properties of the medium (spacetime) in which the object moves, will be further explored, according to the wave-particle equivalence, in what follows.

Wave-particle duality

In the previous sections (the ship- in- the- sea, and the rocket- in- space), we saw the intrinsic relationship between spacetime and the motion of an object in spacetime. In both cases the object is able to move by reducing the energy of the wave of spacetime, gaining thus kinetic energy.

Such an intimate relationship between objects and spacetime, which led Einstein to the mass-energy equivalence (expressed as $E=mc^2$), is described in physics through the notion of wave-particle duality, and the uncertainty principle (which can also be called the complementarity principle).

The momentum p of a wave is expressed by de Broglie's formula

$$p = \frac{h}{\lambda}$$

where h is Planck constant, and λ is the wavelength associated with the wave-like aspects of a material object.

The momentum of a material object (or particle) is given by the common expression

$$p = mv$$

where m is the inertial mass of the object, and v is its speed.

Combining the two previous formulas for the momentum, we take

$$p = mv = \frac{h}{\lambda}$$

This expression relates the properties of a wave (the wavelength λ), to those of a particle (the mass m, and speed v), or of any material object. The constant of proportionality h (Planck's constant), connects quantum physics with classical physics.

The same formula also expresses the momentum-position uncertainty principle. Such an uncertainty can be stated as

$\Delta p \Delta \lambda = h,$

$\Delta p \Delta y \approx \frac{h}{2\pi} = \hbar,$

$\lambda \approx 2\pi y, \qquad \hbar = \frac{h}{2\pi}$

where $\hbar$ is called reduced Planck constant.

The uncertainty in fact has to do with the aspect of complementarity between the wave-like properties of spacetime, and the material-like properties of an object moving in spacetime.

Another fundamental expression in quantum physics, closely related to de Broglie's formula, is Planck energy

$E_P = hf = \frac{hc}{\lambda},$

$c = f\lambda$

where the index '*P*' stands for '*Planck.*' The second formula expresses the speed of the wave *c* as the product of its frequency *f* by its wavelength λ.

From the formula of Planck energy we can take the equivalent expression for the energy-time uncertainty,

$\Delta E \Delta \lambda = hc \Rightarrow$

$\Delta E (c\Delta T) = hc \Rightarrow$

$\Delta E \Delta T = h,$

$\Delta E \Delta t \approx \frac{h}{2\pi} = \hbar,$

$\lambda = cT, \qquad T \approx 2\pi t, \qquad \hbar = \frac{h}{2\pi}$

If we combine Planck energy with de Broglie momentum, we take that

$E_P = \frac{hc}{\lambda} = pc,$

$$p = \frac{h}{\lambda}$$

Strictly speaking, the momentum $p=h/\lambda$ refers to that of a wave, while the momentum $p=mv$ refers to that of an object. But if we literary equate the two expressions, we have

$$E_P = pc = (mv)c = mvc,$$

$$p = \frac{h}{\lambda} = mv$$

Setting $v=c$ in the last expression, we take

$$E_P \equiv E_{Ein} = mc^2$$

It has been such a substitution which led to the confusion between the energy of a wave and the energy of a material object, because the mass m does not refer to a wave, while the energy E_P does not refer to a particle (a material object). Thus the aspect that material objects have wavelike properties was a conclusion based on this misidentification.

This is what made me suspect that in the last expression m does not refer to the mass of the object (the constant of inertia) but to the mass of the *wave*.

Thus here we can make the following distinction between the two momenta. Calling p_m the momentum of the object, and p_μ the momentum of the wave, we have

$$p_m = mv$$

$$p_\mu = \mu c$$

where m will be the mass of the object, while μ will be the mass of the wave.

If we compare these expressions to Newton's second law of motion,

$$\sum F_N = \frac{dp}{dt} = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} = ma + bv, \qquad a = \frac{dv}{dt}, \qquad b = \frac{dm}{dt}$$

where the index '*N*' in F_N stands for '*Newton*,' we see that we can identify two forces, a force F_m, related to the material object, and another force F_μ, related to the wave,

$$F_m = \frac{dp_m}{dt} = m\frac{dv}{dt}, \qquad m = const.$$

$$F_\mu = \frac{dp_\mu}{dt} = \frac{d\mu}{dt}c, \qquad c = const.$$

where m is the inertial mass of the object, and c is the speed of the wave, so that

$$\sum F = F_m + F_\mu = m\frac{dv}{dt} + \frac{d\mu}{dt}c$$

Comparing the expressions for the forces F_m and F_μ, to Newton's formula for the force F_N, we see that

$$v\frac{dm}{dt} = \frac{d\mu}{dt}c$$

This formula can also be written as follows. Substituting,

$$\frac{d\mu}{dt}c = \frac{d\mu}{d\lambda}\frac{d\lambda}{dt}c = \rho c^2, \qquad \rho = \frac{d\mu}{d\lambda}, \qquad \frac{d\lambda}{dt} = c$$

we take

$$v\frac{dm}{dt} = \frac{d\mu}{dt}c \Rightarrow$$

$$bv = \rho c^2,$$

$$b = \frac{dm}{dt}, \qquad \rho = \frac{d\mu}{d\lambda}$$

where ρ is the density of the wave, and b is the damping constant, which we have already seen.

The formula $bv=\rho c^2$ gives us the equivalence between the properties of the wave (μ,c), and those of a material object (m,v), moving in the wave.

Thus the equation of motion of the moving object can be written in two equivalent ways,

$$ma + bv = ma + \rho c^2 = 0,$$

where

$$bv = \rho c^2, \qquad \gamma = \frac{b}{m} = \frac{\rho c^2}{mv}$$

From the equation of motion, we can also derive an equation for the energy of the system. Having,

$$F_m = ma = m\frac{dv}{dt}, \qquad a = \frac{dv}{dt}$$

$$F_\mu = \rho c^2 = \frac{d\mu}{dt}c, \qquad \rho = \frac{d\mu}{dt}, \qquad \frac{d\lambda}{dt} = c$$

we take that,

$$E_m = \int F_m dy = \int m\frac{dv}{dt}dy = m\int \frac{dv}{dt}dy = m\int dv\frac{dy}{dt} = m\int vdv = \frac{1}{2}mv^2$$

$$E_\mu = \int F_\mu d\lambda = \int \frac{d\mu}{dt}cd\lambda = c\int \frac{d\mu}{dt}d\lambda = c\int d\mu\frac{d\lambda}{dt} = c\int d\mu c = c^2\int d\mu = \mu c^2$$

so that

$$\sum E = E_m + E_\mu = \frac{1}{2}mv^2 + \mu c^2 \Rightarrow$$

$$\frac{1}{2}mv^2 + \mu c^2 = \mu_0 c^2$$

where $\mu_0 c^2$ will be the total energy stored in (a region of) spacetime.

This equation for the energy of the system is equivalent to the one we previously saw for the rocket- in- space, where

$$E = E_0 e^{-2\gamma t}$$

$$E_0 = \frac{1}{2}m\gamma^2 y_0^2 = \frac{1}{2}mv_0^2$$

To see this, we may set

$$\mu = \mu_0 e^{-\gamma t}$$

so that

$$\frac{d\mu}{dt} = \frac{d}{dt}(\mu_0 e^{-\gamma t}) = \mu_0 \frac{d}{dt} e^{-\gamma t} = -\gamma \mu_0 e^{-\gamma t} = -\gamma \mu \Rightarrow$$

$$\frac{d\mu}{dt} c^2 = -\gamma \mu c^2 \Rightarrow$$

$$\frac{dE_\mu}{dt} = \frac{d\mu}{dt} c^2 = -\gamma \mu c^2 = -\gamma \mu_0 e^{-\gamma t} \Rightarrow$$

$$E_\mu = E_0 e^{-\gamma t}, \qquad E_0 = \mu_0 c^2$$

The difference in this energy will be the kinetic energy E_m of the object,

$$\Delta E = E_0 - E_\mu = E_0 - E_0 e^{-\gamma t} = E_0(1 - e^{-\gamma t}) = \mu_0 c^2 - \mu c^2 = \frac{1}{2} m v^2 \equiv E_m$$

so that for the total energy we have the equivalent terms,

$$E_0 = m\gamma^2 y_0^2 = m v_0^2 = \mu_0 c^2$$

The factor of ½, which was omitted in the previous expression, is not significant and may disappear if we include a harmonic term in the equation of motion, as we will do later on.

The basic aspect here is that the energy $E_\mu = \mu c^2$ can be identified with the damping energy of the system, (so that the associated force F_μ can also be identified with the damping force). This is the energy stored in the wave of spacetime, and it is transformed by the moving object into its own kinetic energy E_k,

$$E_k \equiv E_m = \Delta\mu c^2 = (\mu_0 - \mu) c^2 = \frac{1}{2} m v^2$$

Therefore the wave-like properties do not refer to the moving material object, but to spacetime. Such a conclusion can only be drawn if we make the distinction between the two masses, the inertial mass m of the object moving in spacetime, and the mass μ of the wave associated with the oscillations of spacetime.

Another remark we can make, is that the expression for the damping energy stored in the wave

$E_\mu = \mu c^2$

is handled much easier than if we tried to calculate the damping energy directly from the following integral,

$$E_d = \int F_d dy = \int bv dy = \int \frac{dm}{dt} v dy = \int dm \frac{dy}{dt} v = \int dm v^2, \qquad v = \frac{dy}{dt}, \qquad b = \frac{dm}{dt}$$

where the index '*d*' stands for 'damping.'

This is also true for the rate of change of the energy

$$\frac{dE_\mu}{dt} = \frac{d\mu}{dt} c^2 = -\gamma \mu c^2$$

Furthermore, comparing the previous expression to the one we have already seen,

$$\frac{dE_k}{dt} = -bv^2 = -\frac{dm}{dt} v^2$$

we take that

$$\frac{dE_k}{dt} \equiv \frac{dE_m}{dt} = -\frac{dE_\mu}{dt} \Rightarrow$$

$$\frac{dm}{dt} v^2 = \frac{d\mu}{dt} c^2$$

This is a second formula of equivalence, additional to the earlier formula

$bv = \rho c^2$

A more accurate description can be made if we separate the coordinate time *t*, related to a clock on the moving object, from the period *T*, referring to the oscillations of the wave. These two times need not be the same, if the object is moving at a speed *v* different from the speed of the wave (presumably the speed of light *c*).

A direct comparison between the times *t* and *T* can be made if we set,

$y = vt$

$$\lambda = cT$$

so that

$$\frac{y}{\lambda} = \frac{v}{c}\frac{t}{T}$$

The coordinate *y* measures the displacement of the moving object on a linear axis *y*, while λ is the displacement of the wave on its curved path.

If the displacement *y* is identified with the amplitude of the wave, then it will simply be $\lambda=2\pi y$. In this case, it will also be

$$\frac{t}{T} = \frac{y}{\lambda}\frac{c}{v} = \frac{y}{2\pi y}\frac{c}{v} = \frac{1}{2\pi}\frac{c}{v} \approx \frac{c}{v}, \qquad \lambda = 2\pi y, \qquad \lambda \approx y$$

so that from the first formula of equivalence $bv=\rho c^2$,

$$bv = \rho c^2 \Rightarrow$$

$$\frac{dm}{dt}v = \frac{d\mu}{d\lambda}c^2 = \frac{d\mu}{dT}\frac{dT}{d\lambda}c^2 = \frac{d\mu}{dT}\frac{1}{c}c^2 = \frac{d\mu}{dT}c, \qquad b = \frac{dm}{dt}, \qquad \rho = \frac{d\mu}{d\lambda}, \qquad c = \frac{d\lambda}{dT} \Rightarrow$$

$$\frac{dm}{dt}v = \frac{d\mu}{dT}c = \frac{d\mu}{dt}\frac{dt}{dT}c = \frac{d\mu}{dt}\frac{c}{v}c = \frac{d\mu}{dt}\frac{c^2}{v}, \qquad \frac{dt}{dT} = \frac{c}{v} \Rightarrow$$

$$\frac{dm}{dt}v^2 = \frac{d\mu}{dt}c^2$$

we take the second formula of equivalence.

The terms in the latter expression may be accompanied by a negative sign, since the energy on the right side of the equation (referring to the energy stored in a region of spacetime) is transformed into the energy on the left side (referring to the kinetic energy of the object moving in the same region).

Significantly, if the speed *v* of the object approaches the speed of light *c*, then the rate at which the wave loses mass μ, will be equal to the rate at which the object loses mass *m* (or burns fuel),

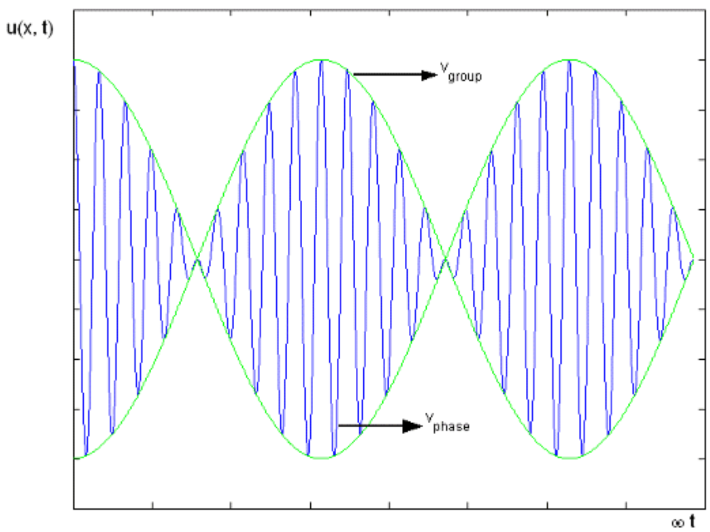
$$\frac{dm}{dt}v^2 = \frac{d\mu}{dt}c^2$$

$$v = c \Rightarrow$$

$$\frac{dm}{dt} = \frac{d\mu}{dt}$$

so that we may suggest that at the speed of light the object will have the best energy efficiency.

Notes:



The envelope (the green line) travels at the group velocity. The carrier wave (the blue line) travels at the phase velocity. Group velocity and phase velocity are not necessarily the same. [http://physics.gmu.edu/~dmaria/590%20Web%20Page/public_html/qm_topics/phase_vel/phase.html]

This is an example of how the introduction of the mass μ of the wave helps us solve the problem of phase velocity. The previous picture illustrates a wave-packet which travels in space. In the context of wave-particle duality, an object of mass m travelling at a speed v will have a momentum $p=mv$, and it will be accompanied by a 'guiding' wave, whose velocity will be the phase velocity $v_p=f\lambda$, f being the frequency, and λ the wavelength of the wave. If we use the formulas

$$p = \frac{h}{\lambda} = mv$$

$$E = hf = \frac{hc}{\lambda} = mc^2$$

for the momentum and the energy, respectively, of a material particle (as if it behaved like a wave), we take that

$$v_p = f\lambda = \frac{E}{h}\frac{h}{p} = \frac{E}{p} = \frac{mc^2}{mv} = \frac{c^2}{v} \Rightarrow$$

$$v_p \geq c, \qquad v \leq c$$

This creates the paradox that the phase velocity can be greater than the speed of light. In quantum physics the paradox is resolved by using the notion of group velocity, which never exceeds the speed of light. Yet the problem of the phase velocity remains, if we confuse the properties of the moving object (m,v) with those of the guiding wave (μ,c). But by acknowledging this distinction, we have that

$$p_\mu = \frac{h}{\lambda} = \mu c$$

$$E_\mu = hf = \frac{hc}{\lambda} = \mu c^2$$

so that the phase velocity will simply be

$$v_p = f\lambda = \frac{E}{h}\frac{h}{p} = \frac{E}{p} = \frac{\mu c^2}{\mu c} = c$$

where the previous formulas for the momentum and the energy strictly refer to a wave.

Thus while the energy $E_\mu = \mu c^2$ refers to the energy of a wave (whose speed c will be that of light), the energy $E_m = mc^2$ refers to the kinetic energy of an object of mass m which, in this case, travels at the speed of light, $v=c$.

The wet sponge example

Here it would be helpful to clarify some points with respect to the meaning of inertia (and the inertial mass m_0).

As we previously saw, if we take Newton's second law of motion, and consider that the (inertial) mass m_0 is a constant, we have

$$\sum F = \frac{dp}{dt} = \frac{d}{dt}(m_0 v) = m_0 \frac{dv}{dt} = m_0 a(t), \qquad a(t) = \frac{dv}{dt}, \qquad m_0 = const.$$

But if the mass changes, then we have that,

$$\sum F = \frac{dp}{dt} = \frac{d}{dt}(mv) = m_0 \frac{dv}{dt} + v_0 \frac{dm}{dt} = m_0 a(t) + b(t) v_0,$$

$$a(t) = \frac{dv}{dt}, \qquad b(t) = \frac{dm(t)}{dt}, \qquad m_0 = const., \qquad v_0 = const.$$

The quantities m_0 (presumably the inertial mass) and v_0 (for example the final speed) will have to be determined by some boundary conditions. Therefore it has to be justified whether the inertial mass m_0 refers to the initial (total) mass of the object in motion, or to its final mass, as well as what is the difference between the inertial mass m_0, which is constant, and the mass $m(t)$, which changes.

For this purpose we will use the example of the wet sponge.

So let's imagine that we have a tank filled with water. We will call the mass of the water μ_0. We also have a sponge, which initially is dry, so that its initial mass is m_0. Supposedly this will be the sponge's inertial mass. Then we sink the sponge into the tank of water, so that the sponge absorbs a quantity of water. If we call m_f the mass of the absorbed water, then the total mass m of the sponge will be $m=m_0+m_f$.

$$m = m_0 + m_f$$

The problem here is that we want to measure the inertial mass m_0 of the sponge. The way to do this is in two steps. First we measure the mass m of the sponge filled with water. Then we drain

the water off the sponge, and measure the mass of the drained water, thus the mass m_f of the absorbed water. Therefore the inertial mass of the sponge will be the difference,

$$m_0 = m - m_f$$

One might wonder why we didn't measure the inertial mass m_0 of the sponge (e.g. by weighing it), in the first place. The answer is that we cannot directly measure the mass of the sponge *if we use the mass of the water as the reference unit of measurement*.

Another way to put this, is to say that we cannot measure the inertial mass m_0 of an object in free fall, because, from the equation of motion $m_0a=m_0g$, the inertial mass disappears. This is to say that the falling object is embodied in the gravitational field (the 'water') which makes the object fall.

Therefore we may say that the inertial mass m_0 of the sponge (or of any object) is what is left after all the other masses (the 'water') have been excluded.

Now if we call M_0 the total mass of the water together with the sponge, we can write the following equation for the masses,

$$M_0 = \mu + m = \mu + m_0 + m_f = \mu_0 + m_0$$

$$m = m_0 + m_f$$

$$\mu = \mu_0 - m_f$$

where μ is the mass of water remaining in the tank, after a quantity m_f of water has been absorbed by the sponge.

Here, in order to return to the case of a rocket in space, we can identify the mass μ_0 with the total mass in a region of spacetime, m_f will be the amount of fuel of the spaceship (the amount of energy the spaceship has absorbed in the form of fuel), m_0 will be the spaceship's inertial mass (its mass without the fuel), while μ will be the mass left to spacetime.

The rate of change of the mass m of the spaceship will be,

$$m = m_0 + m_f \Rightarrow$$

$$\frac{dm}{dt} = \frac{d}{dt}\left(m_0 + m_f\right) = \frac{dm_f}{dt}, \qquad m_0 = const.$$

Thus what changes is not the inertial mass m_0 of the spaceship, but the mass m_f of the fuel it contains. In such a sense, the total mass m of the spaceship is reduced, as the spaceship burns fuel.

If the spaceship burns all its available fuel m_f, then it will be left with its inertial mass m_0,

$$m = m_0 + m_f,$$

$$m_f \to 0 \Rightarrow$$

$$m \equiv m_0$$

On the other hand, the rate of change of the mass μ of spacetime will be,

$$\mu = \mu_0 - m_f \Rightarrow$$

$$\frac{d\mu}{dt} = \frac{d}{dt}\left(\mu_0 - m_f\right) = -\frac{dm_f}{dt}, \qquad \mu_0 = const.$$

Thus spacetime loses mass μ to the degree the spaceship burns fuel m_f. If the spaceship makes use of all the available energy stored in a region of spacetime, then the mass μ_0 of spacetime will have been transformed into the fuel m_f of the spaceship,

$$\mu = \mu_0 - m_f$$

$$\mu \to 0 \Rightarrow$$

$$\mu_0 = m_f$$

Thus the total mass m_f of fuel the spaceship consumes cannot be greater than the total mass μ_0 available in a region of spacetime.

Notes:

The previous conclusion may be shown graphically in the following way. Firstly, we need to determine if the damping factor γ,

$$\gamma = \frac{b}{m} = \frac{dm}{dt}\frac{1}{m}, \qquad b = \frac{dm}{dt}$$

is constant, or if it changes with time.

The simplest case is to assume that all related quantities are constant, so that the damping factor will also be constant,

$$\gamma_0 = \frac{b_0}{m_0} = \frac{dm}{dt}\frac{1}{m_0} = const., \qquad b_0 = \frac{dm}{dt} = const., \qquad m_0 = const.$$

In that case, the damping factor will be defined with respect to the inertial mass m_0.

Still, we are left with a mass *m(t)* which changes with time,

$$\gamma_0 = \frac{dm}{dt}\frac{1}{m_0} \Rightarrow$$

$$\frac{dm}{dt} = m_0\gamma_0 \Rightarrow$$

$$\int \frac{dm}{dt}dt = \int m_0\gamma_0 dt \Rightarrow$$

$$\int dm = m_0\gamma_0 \int dt\,, \qquad m_0 = const., \qquad \gamma_0 = const. \Rightarrow$$

$$m(t) = m_0\gamma_0 t$$

Another way to keep the damping factor constant, is to keep the rate *b(t)/m(t)* constant,

$$\gamma_0 = \frac{b(t)}{m(t)} = \frac{dm(t)}{dt}\frac{1}{m(t)} = const., \qquad b(t) = \frac{dm(t)}{dt}$$

so that for the mass *m(t)* we take that,

$$\gamma_0 = \frac{dm(t)}{dt}\frac{1}{m(t)} \Rightarrow$$

$$\frac{dm(t)}{m(t)} = \gamma_0 dt \Rightarrow$$

$$\int \frac{dm(t)}{m(t)} = \int \gamma_0 dt \Rightarrow$$

$$\int \frac{dm(t)}{m(t)} = \gamma_0 \int dt\,, \qquad \gamma_0 = const. \Rightarrow$$

$$\ln m(t) = \gamma_0 t + C \Rightarrow$$

$$\ln m(t) = \gamma_0 t + \ln m_0\,, \qquad C = \ln m_0\,, \qquad m_0 = e^C \Rightarrow$$

$$\ln m(t) - \ln m_0 = \ln \frac{m(t)}{m_0} = \gamma_0 t \Rightarrow$$

$$\frac{m(t)}{m_0} = e^{\gamma_0 t} \Rightarrow$$

$$m(t) = m_0 e^{\gamma_0 t}$$

In this expression, since the mass *m(t)* increases exponentially with time, the quantity m_0 will be the initial minimum mass (presumably the inertial mass).

If we assume that the mass *m(t)* decreases with time, then we may use a negative sign in the exponential, so that the quantity m_0 will be the maximum mass.

However, in the general case, if we consider a damping factor *γ(t)* which changes with time,

$$\gamma(t) = \frac{b(t)}{m(t)} = \frac{dm(t)}{dt}\frac{1}{m(t)}, \qquad b(t) = \frac{dm(t)}{dt}$$

then with respect to the mass *m(t)* we take that,

$$\gamma(t) = \frac{dm(t)}{dt}\frac{1}{m(t)} \Rightarrow$$

$$\frac{dm(t)}{m(t)} = \gamma(t)dt \Rightarrow$$

$$\int \frac{dm(t)}{m(t)} = \int \gamma(t)dt$$

In order to deal with the integral on the right hand side of the previous equation, we may simply set

$$\gamma(t) = \delta t, \qquad \delta = const.$$

so that the last expression takes the form

$$\int \frac{dm(t)}{m(t)} = \int \gamma(t)dt = \int \delta t dt = \delta \int t dt \Rightarrow$$

$$\int_{m(t)}^{m_0} \frac{dm(t)}{m(t)} = \delta \int_t^{t_0} t dt \Rightarrow$$

$$\ln m_0 - \ln m(t) = \delta \left(\frac{t_0^2}{2} - \frac{t^2}{2} \right) \Rightarrow$$

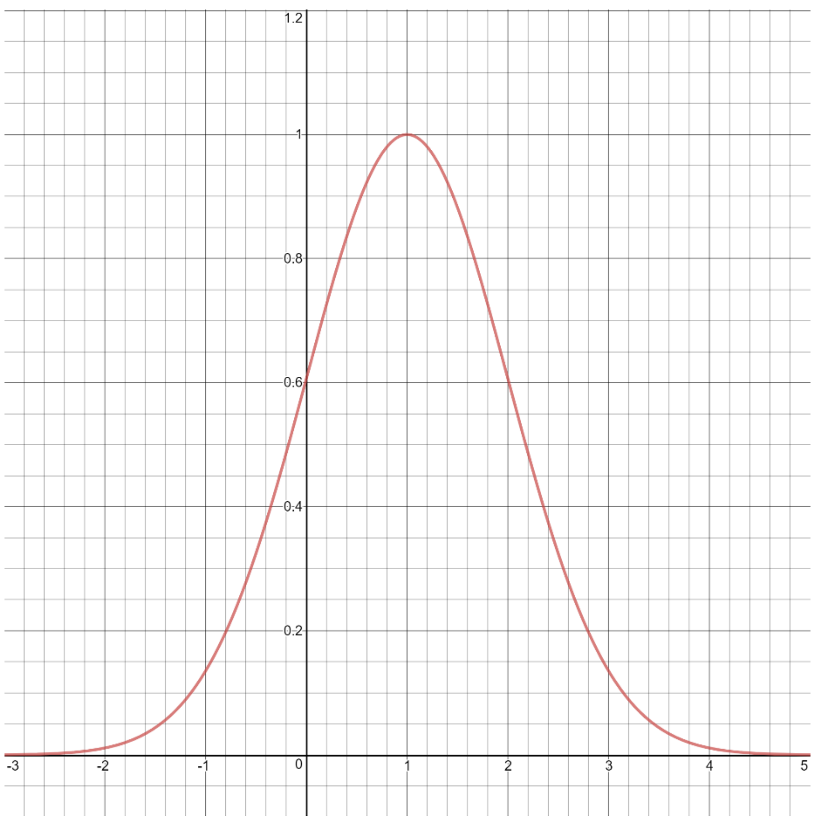
$$\ln \frac{m_0}{m(t)} = \frac{1}{2} \delta (t_0^2 - t^2) \Rightarrow$$

$$\ln \frac{m(t)}{m_0} = -\frac{1}{2} \delta (t_0^2 - t^2) \Rightarrow$$

$$\frac{m(t)}{m_0} = e^{-\frac{1}{2}\delta(t_0^2 - t^2)} \Rightarrow$$

$$m(t) = m_0 e^{-\frac{1}{2}\delta(t_0^2 - t^2)}$$

This is the related graph:



In this graph we have set $t_0 = 1$, so that the function

$$m(t) = m_0 e^{-\delta\frac{(t_0 - t)^2}{2}} = m_0 e^{-\frac{1}{2}\delta(1-t)^2}, \qquad t_0 = 1$$

takes its maximum value m_0 at some boundary time $t=t_0$.

This m_0 will refer to the maximum mass available to the object, thus it can also be identified with the total mass μ_0 of spacetime. Therefore if we replace m_0 by μ_0, and also set

$$\gamma(t)t = (\delta t)t = \delta t^2 = n^2 = \frac{v^2}{c^2}$$

we take

$$m(t) = m_0 e^{-\delta\frac{(t_0-t)^2}{2}} \Rightarrow$$

$$m(t) = \mu_0 e^{-\delta\frac{(t_0-t)^2}{2}}, \qquad m_0 \equiv \mu_0 \Rightarrow$$

$$m(t) = \mu_0 e^{-e^{-\frac{1}{2}\frac{\left(v_0^2-v\right)^2}{c^2}}}, \qquad \delta(t_0^2 - t^2) = \frac{v_0^2 - v^2}{c^2}$$

If additionally we set $v_0=c$, so that the function takes its maximum value μ_0 at $v=c$, we take

$$m(t) = \mu_0 e^{-\frac{1}{2}\frac{(v_0-v)^2}{c^2}} \Rightarrow$$

$$m(t) = \mu_0 e^{-\frac{1}{2}\frac{(c-v)^2}{c^2}}, \qquad v_0 = c \Rightarrow$$

$$m(t) = \mu_0 e^{-\frac{1}{2}(1-v)^2}, \qquad c = 1$$

Such transformations are indicative of the fact that the mass $m(t)$ of the object, although it takes its maximum value μ_0 at the speed of light, $v=c$, can still be defined even if the speed of the object is greater than the speed of light, $v>c$. This can happen if the mass μ_0 of the wave of spacetime is a mass ‘per oscillation,’ or per wavelength of the wave, so that there is more mass, thus also energy, available as fuel to the object moving in spacetime.

The damped harmonic oscillator

Up till now we have seen two examples of motion, that of the ship- in- the- sea, and that of the rocket-in-space. The equations of motion are respectively

$$ma + ky = 0$$

$$ma + bv = 0$$

If we combine the two equations (without repeating the term *ma*) we take

$$ma + ky + bv = 0$$

This is the equation of the damped harmonic oscillator. It includes both a harmonic term (the elastic force $F_{el}=ky$), and a damping term (the damping force $F_d=bv$).

Rewriting the equation of motion as,

$$ma + ky + bv = 0 \Rightarrow$$

$$a + \frac{k}{m}y + \frac{b}{m}v = 0 \Rightarrow$$

$$a + \omega_0^2 y + \gamma v = 0,$$

$$a = \frac{dv}{dt}, \qquad \omega_0^2 = \frac{k}{m}, \qquad \gamma = \frac{b}{m}, \qquad b = \frac{dm}{dt}$$

we can identify the quantity γ with the damping factor (so that we may call *b* the damping constant, and γ the damping factor). The quantity ω_0 is the angular frequency of the simple harmonic oscillator (without damping).

Here we will assume a possible solution for the equation of motion of the damped harmonic oscillator, in the following form,

$$ma + ky + bv = 0,$$

$$y(t) = y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi)$$

where ω' is the angular frequency of the damped oscillator (different from the angular frequency ω_0), while φ is the phase, and its value depends on the boundary conditions.

For the speed v and the acceleration a, we will have

$$y(t) = y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) \Rightarrow$$

$$v(t) = \frac{dy(t)}{dt} = -\frac{\gamma}{2} y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) - \omega' y_0 e^{-\frac{\gamma}{2}t}\sin(\omega' t - \varphi)$$

$$a(t) = \frac{dv(t)}{dt} = \frac{\gamma^2}{4} y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) + \frac{\gamma}{2}\omega' y_0 e^{-\frac{\gamma}{2}t}\sin(\omega' t - \varphi)$$

$$+\frac{\gamma}{2}\omega' y_0 e^{-\frac{\gamma}{2}t}\sin(\omega' t - \varphi) - \omega'^2 y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) \Rightarrow$$

$$a(t) = \frac{\gamma^2}{4} y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) + \gamma\omega' y_0 e^{-\frac{\gamma}{2}t}\sin(\omega' t - \varphi) - \omega'^2 y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi)$$

Substituting this solution into the equation of motion, we take

$$ma + bv + ky = 0, \qquad a + \gamma v + \omega_0^2 y = 0, \qquad \gamma = \frac{b}{m}, \qquad \omega_0^2 = \frac{k}{m} \Rightarrow$$

$$\frac{\gamma^2}{4} y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) + \gamma\omega' y_0 e^{-\frac{\gamma}{2}t}\sin(\omega' t - \varphi) - \omega'^2 y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi)$$

$$-\frac{\gamma^2}{2} y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) - \gamma\omega' y_0 e^{-\frac{\gamma}{2}t}\sin(\omega' t - \varphi) + \omega_0^2 y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) = 0 \Rightarrow$$

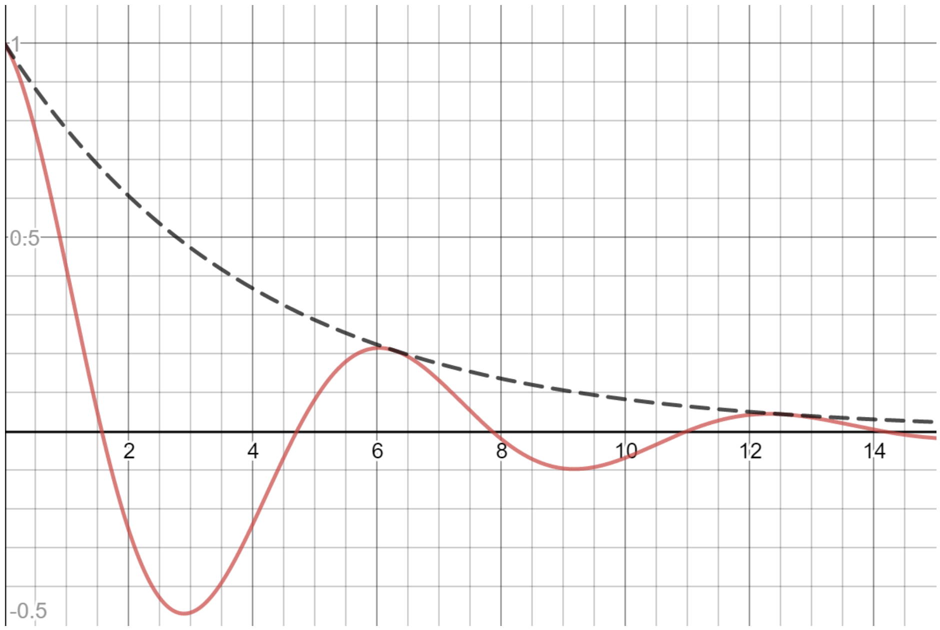
$$-\frac{\gamma^2}{4} y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) - \omega'^2 y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) + \omega_0^2 y_0 e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) = 0 \Rightarrow$$

$$-\frac{\gamma^2}{4} - \omega'^2 + \omega_0^2 = 0 \Rightarrow$$

$$\omega'^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

The last equation is a condition which satisfies the given solution.

The graph of this solution is the following one,



Red: $y(t) = y_0 e^{-\frac{\gamma}{2}t} \cos \omega' t \approx e^{-\frac{1}{4}t} \cos t\,,\ \varphi = 0$

Black (dotted): $y(t) = y_0 e^{-\frac{\gamma}{2}t} \approx e^{-\frac{1}{4}}$

$$y_0 = 1, \qquad \gamma = \frac{1}{2}, \qquad \omega_0 = 1, \qquad \omega'^2 = \omega_0^2 - \frac{\gamma^2}{4} = 1 - \frac{1}{16} \approx 1$$

The black dotted line in the previous graph is the exponential envelope of the amplitude (red line). Thus the amplitude *y(t)* oscillates because of the cosine term, and also decays because of the exponential term.

With respect to the energy of the system, the elastic energy E_{el} and the kinetic energy E_k will be commonly given by the following expressions,

$$ma + bv + ky = 0,$$

$$y(t) = y_0 e^{-\frac{\gamma}{2}t} \cos \omega' t\,, \qquad \varphi = 0$$

$$v(t) = -\frac{\gamma}{2} y_0 e^{-\frac{\gamma}{2}t} \cos \omega' t - \omega' y_0 e^{-\frac{\gamma}{2}t} \sin \omega' t,$$

so that

$$E_{el}(t) = \frac{1}{2} k y^2(t) = \frac{1}{2} m \omega_0^2 y_0^2 e^{-\gamma t} \cos^2 \omega' t$$

$$E_k(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}m\left(-\frac{\gamma}{2}y_0e^{-\frac{\gamma}{2}t}\cos\omega' t - \omega' y_0e^{-\frac{\gamma}{2}t}\sin\omega' t\right)^2 \Rightarrow$$

$$E_k(t) = \frac{1}{2}me^{-\gamma t}\left(-\frac{\gamma}{2}y_0\cos\omega' t - \omega' y_0\sin\omega' t\right)^2$$

The expression for the kinetic energy E_k is somewhat complicated. Choosing however some initial condition,

$$t = 0, \qquad y(0) = y_0, \qquad v(0) = -\frac{\gamma}{2}y_0 = -v_0 \Rightarrow$$

$$E_{el}(0) = \frac{1}{2}ky^2(0) = \frac{1}{2}m\omega_0^2y_0^2, \qquad k = m\omega_0^2$$

$$E_k(0) = \frac{1}{2}mv^2(0) = \frac{1}{2}m\frac{\gamma^2}{4}y_0^2 = \frac{1}{2}mv_0^2, \qquad v_0 = \frac{\gamma}{2}y_0$$

we take simpler expressions for the elastic energy E_{el} and the kinetic energy E_k (in the form of the total energy).

Another boundary condition can be taken at some time t equal to the period T of the oscillator,

$$t = T, \qquad \varphi = 0 \Rightarrow$$

$$\omega' T = \frac{2\pi}{T}T = 2\pi,, \qquad v(T) = -\frac{\gamma}{2}y_0e^{-\frac{\gamma}{2}T} \Rightarrow$$

$$\cos(\omega' t - \varphi) = \cos\omega' T = \cos 2\pi = 1,$$

$$\sin(\omega' t - \varphi) = \sin\omega' T = \sin 2\pi = 0,$$

$$y(t) = y_0e^{-\frac{\gamma}{2}}\cos(\omega' t - \varphi) \Rightarrow$$

$$y(T) = y_0e^{-\frac{\gamma}{2}T},$$

$$v(t) = -\frac{\gamma}{2}y_0e^{-\frac{\gamma}{2}t}\cos(\omega' t - \varphi) - \omega' y_0e^{-\frac{\gamma}{2}t}\sin(\omega' t - \varphi) \Rightarrow$$

$$v(T) = -\frac{\gamma}{2}y_0e^{-\frac{\gamma}{2}T}$$

so that at the conclusion of a cycle, t=T, for the elastic energy E_{el}, and the kinetic energy E_k, we have

$$E_{el}(T) = \frac{1}{2}ky^2(T) = \frac{1}{2}m\omega_0^2y_0^2e^{-\gamma T}$$

$$E_k(T) = \frac{1}{2}mv^2(T) = \frac{1}{2}m\frac{\gamma^2}{4}y_0^2e^{-\gamma T}$$

The mechanical energy E_m of the system at times $t=0$ and $t=T$ will respectively be,

$$E_m(0) = E_{el}(0) + E_k(0) = \frac{1}{2}m\omega_0^2y_0^2 + \frac{1}{2}m\frac{\gamma^2}{4}y_0^2 = \frac{1}{2}m\left(\omega_0^2 + \frac{\gamma^2}{4}\right)y_0^2$$

$$E_m(T) = E_{el}(T) + E_k(T) = \frac{1}{2}m\omega_0^2y_0^2e^{-\gamma T} + \frac{1}{2}m\frac{\gamma^2}{4}y_0^2e^{-\gamma T} = \frac{1}{2}m\left(\omega_0^2 + \frac{\gamma^2}{4}\right)y_0^2e^{-\gamma T}$$

Thus at the conclusion of one cycle the change of the mechanical energy will be,

$$\Delta E_m = E_m(T) - E_m(0) = \frac{1}{2}m\left(\omega_0^2 + \frac{\gamma^2}{4}\right)y_0^2e^{-\gamma T} - \frac{1}{2}m\left(\omega_0^2 + \frac{\gamma^2}{4}\right)y_0^2 \Rightarrow$$

$$\Delta E_m = \frac{1}{2}m\left(\omega_0^2 + \frac{\gamma^2}{4}\right)y_0^2(e^{-\gamma T} - 1) = E_0(e^{-\gamma T} - 1) = -E_0(1 - e^{-\gamma T}),$$

$$E_0 = \frac{1}{2}m\left(\omega_0^2 + \frac{\gamma^2}{4}\right)y_0^2$$

Here the total mechanical energy $E_0 \equiv E_m(t=0)$ includes both an elastic term $\frac{1}{2}m\omega_0^2y_0^2$, and a damping term $\frac{1}{2}m(\gamma^2/4)y_0^2$.

The final boundary condition at time $t=T$ also defines the mechanical energy E_m of the oscillator at any time t. Setting for example $t=\nu T$, where ν is a positive integer, we have that

$$\omega' t = \frac{2\pi}{T}t = \frac{2\pi}{T}\nu t = 2\nu\pi, \qquad \nu = [0,1,2,\ldots]$$

$$\cos\omega' t = \cos 2\nu\pi = 1, \qquad \varphi = 0$$

$$\sin\omega' t = \sin 2\nu\pi = 0,$$

$$y(t) = y_0e^{-\frac{\gamma}{2}t}$$

$$v(t) = -\frac{\gamma}{2}y_0e^{-\frac{\gamma}{2}t},$$

$$a(t) = \frac{\gamma^2}{4}y_0e^{-\frac{\gamma}{2}t} - \omega'^2y_0e^{-\frac{\gamma}{2}t}$$

so that the expression for the mechanical energy can be written as

$$E_m(t) = E_{el}(t) + E_k(t) = \frac{1}{2}ky^2(t) + \frac{1}{2}mv^2(t) = \frac{1}{2}m\omega_0^2 y_0^2 e^{-\gamma t} + \frac{1}{2}m\frac{\gamma^2}{4}y_0^2 e^{-\gamma t} \Rightarrow$$

$$E_m(t) = \frac{1}{2}m\left(\omega_0^2 + \frac{\gamma^2}{4}\right)y_0^2 e^{-\gamma t} = E_0 e^{-\gamma t}, \qquad E_0 = \frac{1}{2}m\left(\omega_0^2 + \frac{\gamma^2}{4}\right)y_0^2$$

The same solution for the mechanical energy can be taken in a straightforward way, if we consider the rate of change of the mechanical energy. Multiplying each term in the equation of motion by v, we take

$$ma + ky + bv = 0 \Rightarrow$$

$$mav + kyv + bv^2 = 0 \Rightarrow$$

$$mav + kyv = -bv^2$$

The first term is the rate of change of the mechanical energy,

$$\frac{dE_m(t)}{dt} = \frac{dE_{el}(t)}{dt} + \frac{dE_k(t)}{dt} = \frac{d}{dt}\left[\frac{1}{2}ky^2(t)\right] + \frac{d}{dt}\left[\frac{1}{2}mv^2(t)\right] \Rightarrow$$

$$\frac{dE_m(t)}{dt} = \frac{1}{2}k\frac{d}{dt}[y^2(t)] + \frac{1}{2}m\frac{d}{dt}[v^2(t)] = \frac{1}{2}k\left[2y(t)\frac{dy(t)}{dt}\right] + \frac{1}{2}m\left[2v(t)\frac{dv(t)}{dt}\right] \Rightarrow$$

$$\frac{dE_m}{dt}(t) = ky(t)v(t) + mv(t)a(t)$$

Therefore the second term can be identified with the rate of change of the damping energy,

$$mav + kyv = -bv^2 \Rightarrow$$

$$\frac{dE_m(t)}{dt} = -\frac{dE_d(t)}{dt} = -bv^2(t)$$

The last term can be compared to the kinetic energy,

$$\frac{dE_m(t)}{dt} = -bv^2(t) = -bv^2(t)\frac{E_k(t)}{\frac{1}{2}mv^2(t)} = -\frac{2b}{m}E_k(t) = -2\gamma E_k(t), \qquad \gamma = \frac{b}{m}$$

We may assume that the average rate of change of the kinetic energy is half the rate of change of the mechanical energy. This is based on the simple harmonic oscillator (without damping), where for the energies

$$E_{el}(t) = \frac{1}{2}ky^2(t) = \frac{1}{2}m\omega_0^2 y_0^2 \cos^2 \omega_0 t,$$

$$E_k(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}mv_0^2 \sin^2 \omega_0 t,$$

$$E_m(t) = E_{el}(t) + E_k(t) = \frac{1}{2}m\omega_0^2 y_0^2 \cos^2 \omega_0 t + \frac{1}{2}mv_0^2 \sin^2 \omega_0 t \Rightarrow$$

$$E_m(t) = \frac{1}{2}mv_0^2 \cos^2 \omega_0 t + \frac{1}{2}mv_0^2 \sin^2 \omega_0 t = \frac{1}{2}mv_0^2(\cos^2 \omega_0 t + \sin^2 \omega_0 t) = \frac{1}{2}mv_0^2,$$

$$v_0 = \omega_0 y_0,$$

we have the following average values,

$$\langle E_{el}\rangle = \langle\frac{1}{2}m\omega_0^2 y_0^2 \cos^2 \omega_0 t\rangle = \frac{1}{2}m\omega_0^2 y_0^2 \langle\cos^2 \omega_0 t\rangle = \frac{1}{4}\omega_0^2 y_0^2, \qquad \langle\cos^2 \omega_0 t\rangle = \frac{1}{2}$$

$$\langle E_k\rangle = \langle\frac{1}{2}mv_0^2 \sin^2 \omega_0 t\rangle = \frac{1}{2}mv_0^2\langle\sin^2 \omega_0 t\rangle = \frac{1}{4}mv_0^2, \qquad \langle\sin^2 \omega_0 t\rangle = \frac{1}{2}$$

$$\langle E_m\rangle = \langle E_{el}\rangle + \langle E_k\rangle = \frac{1}{4}\omega_0^2 y_0^2 + \frac{1}{4}mv_0^2 = \frac{1}{4}mv_0^2 + \frac{1}{4}mv_0^2 = \frac{1}{2}mv_0^2, \qquad v_0 = \omega_0 y_0 \Rightarrow$$

$$\langle E_m\rangle = 2\langle E_k\rangle$$

That the average value of $sin^2\omega_0 t$, or of $cos^2\omega_0 t$, is ½ will be calculated later on.

Thus the previous equation takes the form,

$$\frac{dE_m(t)}{dt} = -\frac{dE_d(t)}{dt} = -bv^2(t) = -2\gamma E_k(t) = -\gamma E_m(t) \Rightarrow$$

$$\frac{dE_m(t)}{E_m(t)} = -\gamma dt \Rightarrow$$

$$\int \frac{dE_m(t)}{E_m(t)} = \int -\gamma dt = -\gamma \int dt \Rightarrow$$

$$\ln E_m(t) = -\gamma t + C \Rightarrow$$

$$E_m(t) = e^{-\gamma t + C} = e^{-\gamma t}e^C = E_0 e^{-\gamma t}, \qquad E_0 = e^C$$

Thus we have the following equations, with respect to the energy of the damped harmonic oscillator,

$$E_m(t) = E_0 e^{-\gamma t},$$

$$\Delta E_m(t) = E_m(t) - E_0 = E_0 e^{-\gamma t} - E_0 = E_0(e^{-\gamma t} - 1) = -E_0(1 - e^{-\gamma t}),$$

$$E_0 = \frac{1}{2} m \left(\omega_0^2 + \frac{\gamma^2}{4}\right) y_0^2 = \mu_0 c^2$$

This result is similar to the one we took in the rocket- in- space example, with the exception of the factor *2* in the exponential, which here is absent because of the contribution of the harmonic (the elastic) term *ky*.

The term $\mu_0 c^2$, which was included in the last expression for the total energy E_0 of the oscillator, refers to the mass μ_0 of the oscillator, not to the mass *m* of the object (‘hanging’ on the ‘spring’ of the oscillator).

Critical damping

An interesting case arises if the damping angular frequency ω' is equal to zero (critical damping). Taking the condition,

$$\omega'^2 = \omega_0^2 - \frac{\gamma^2}{4}$$

and setting $\omega'=0$, we have,

$$\omega' = 0 \Rightarrow$$

$$\omega_0 = \frac{\gamma}{2}, \qquad \omega_0^2 = \frac{\gamma^2}{4} = \frac{k}{m}$$

so that from the equation of motion

$$ma + ky + bv = 0$$

we have the following solution,

$$y(t) = y_0 e^{-\frac{\gamma}{2}t} \cos\omega' t,$$

$$\omega' = 0, \qquad \varphi = 0 \Rightarrow$$

$$y(t) = y_0 e^{-\frac{\gamma}{2}t},$$

$$v(t) = -y_0 \omega' e^{-\frac{\gamma}{2}t} \sin\omega' t - y_0 \frac{\gamma}{2} e^{-\frac{\gamma}{2}t} \cos\omega' t \Rightarrow$$

$$v(t) = -y_0 \frac{\gamma}{2} e^{-\frac{\gamma}{2}t} = -\frac{\gamma}{2} y(t) = -v_0 e^{-\frac{\gamma}{2}t}, \qquad v_0 = \frac{\gamma}{2} y_0 = \omega_0 y_0,$$

$$a(t) = -y_0 \omega'^2 e^{-\frac{\gamma}{2}t} \cos\omega' t + 2y_0 \frac{\gamma}{2} \omega' e^{-\frac{\gamma}{2}t} \sin\omega' t + y_0 \frac{\gamma^2}{4} e^{-\frac{\gamma}{2}t} \cos\omega' t \Rightarrow$$

$$a(t) = y_0 \frac{\gamma^2}{4} e^{-\frac{\gamma}{2}t} = \frac{\gamma^2}{4} y(t) = -\frac{\gamma}{2} v(t) = a_0 e^{-\frac{\gamma}{2}t}, \qquad a_0 = \frac{\gamma^2}{4} y_0 = \frac{\gamma}{2} v_0 = \omega_0^2 y_0,$$

$$\omega' = 0, \qquad \cos\omega' t = 1, \qquad \sin\omega' t = 0$$

At time $t=0$, we have

$$y(0) = y_0,$$

$$v(0) = -v_0 = -\frac{\gamma}{2} y_0,$$

$$a(0) = \frac{\gamma^2}{4} y_0 = \frac{\gamma}{2} v_0 = a_0$$

Accordingly, the energies at time *t=0* will be,

$$E_{el}(0) = \frac{1}{2} k y^2(0) = \frac{1}{2} m\omega_0^2 y_0^2$$

$$E_k(0) = \frac{1}{2} m v^2(0) = \frac{1}{2} m \frac{\gamma^2}{4} y_0^2 = \frac{1}{2} m \frac{4\omega_0^2}{4} y_0^2 = \frac{1}{2} m\omega_0^2 y_0^2 = \frac{1}{2} m v_0^2 = E_{el}(0)$$

$$E_m(0) = E_{el}(0) + E_k(0) = 2E_{el}(0) = 2E_k(0) = m\omega_0^2 y_0^2 = m \frac{\gamma^2}{4} y_0^2 = m v_0^2$$

Therefore, the total energy will be

$$E_0 = m v_0^2 = m \frac{\gamma^2}{4} y_0^2 = m\omega_0^2 y_0^2 = \mu_0 c^2$$

where the last term refers to the mass μ_0 of the oscillator.

We may notice that if the final speed v_0 of the object ('hanging' from the oscillator) is equal to the speed of light *c* (assuming that this is the speed of the oscillator) then it will be

$$v_0 = c \Rightarrow$$

$$m v_0^2 = m c^2 = \mu_0 c^2 \Rightarrow$$

$$m = \mu_0$$

so that the moving object will have consumed a mass equal to the mass per wavelength of the oscillator, as soon as it reaches the speed of light.

At any time *t*, the energies will be

$$E_{el}(t) = \frac{1}{2} k y^2(t) = \frac{1}{2} m\omega_0^2 y_0^2 e^{-\gamma t}$$

$$E_k(t) = \frac{1}{2} m v^2(t) = \frac{1}{2} m v_0^2 e^{-\gamma t} = \frac{1}{2} m \frac{\gamma^2}{4} y_0^2 e^{-\gamma t} = \frac{1}{2} m\omega_0^2 y_0^2 e^{-\gamma t} = E_{el}(t)$$

$$E_m(t) = E_{el}(t) + E_k(t) = 2E_{el}(t) = 2E_k(t) \Rightarrow$$

$$E_m(t) = m\omega_0^2 y_0^2 e^{-\gamma t} = m\frac{\gamma^2}{4} y_0^2 e^{-\gamma t} = m v_0^2 e^{-\gamma t} = \mu c^2$$

The last equation can be written simply as

$E(t) = E_0 e^{-\gamma t}$,

$$E_0 = m v_0^2 = m\frac{\gamma^2}{4} y_0^2 = m\omega_0^2 y_0^2 = \mu_0 c^2$$

Thus when critical damping is applied, the energy of the system is unified by a common expression for the different forms of energies.

Notes:

This is a comparison between the energy of the damped oscillator, and the energy in relativity.

In the ‘wet sponge’ example, we saw that if the damping factor γ changes with time,

$\gamma(t) = \delta t, \qquad \delta = const.$,

then we take for the mass of the object an expression of the form

$$m(t) = m_0 e^{-\delta\frac{t^2}{2}}$$

Here we can make a similar assumption with respect to the energy of the damped oscillator,

$E(t) = E_0 e^{-\gamma t}$,

so that setting, for example,

$$\gamma(t) = \frac{1}{2}\delta t,$$

$$\gamma t = \frac{1}{2}\delta t^2 = \frac{1}{2}n^2 = \frac{1}{2}\frac{v^2}{c^2},$$

$$n = \frac{v}{c}$$

we have for the energy that

$E(n) = E_0 e^{-\frac{1}{2}n^2}$,

$E(v) = E_0 e^{-\frac{1v^2}{2c^2}}$

In this equation, the energy has the form of a Gaussian function.

If we raise both terms to the power of two (in order to make the ½ factor disappear), we have

$E(v) = E_0 e^{-\frac{1v^2}{2c^2}} \Rightarrow$

$[E(v)]^2 = E_0^2 e^{-\frac{v^2}{c^2}}$

The linear (or better the second order) approximation of the previous exponential, at *v<<c*, *v≈0*, is

$e^{-\frac{v^2}{c^2}} \approx 1 - \frac{v^2}{c^2}$

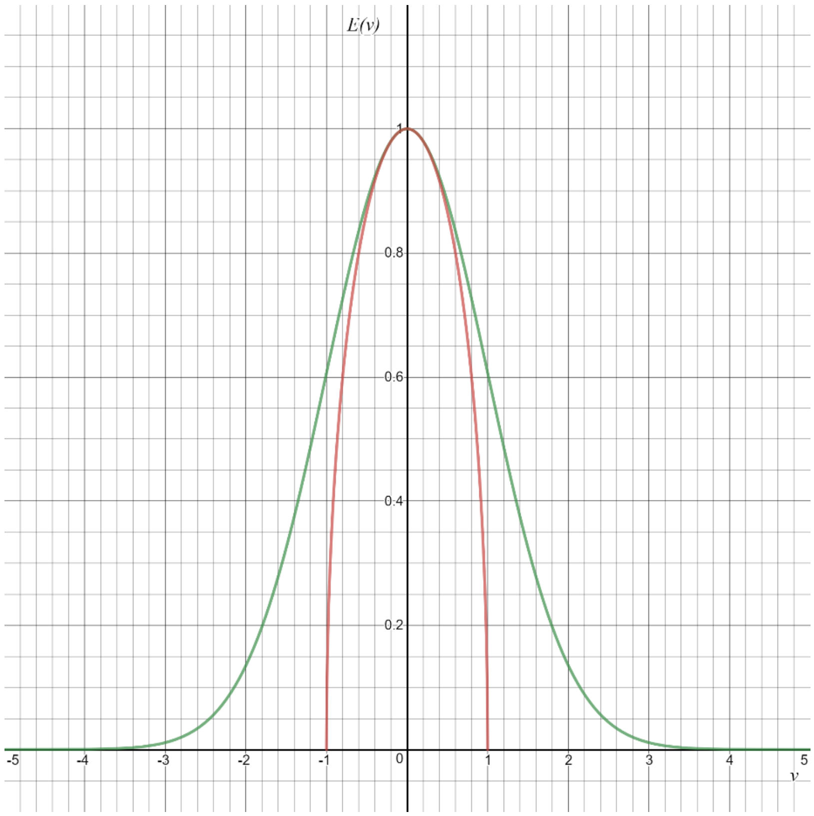
so that

$[E(v)]^2 = E_0^2 e^{-\frac{v^2}{c^2}} \approx E_0^2 \left(1 - \frac{v^2}{c^2}\right) \Rightarrow$

$E(v) \approx E_0 \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma_L} E_0$

and we take back the relativistic expression for the energy.

This is a related graph:



The red line is the relativistic energy in the form of

$$E(v) = \frac{1}{\gamma_L} E_0 = E_0 \sqrt{1 - \frac{v^2}{c^2}} \approx \sqrt{1 - v^2}$$

while the green line is the exponential energy in the Gaussian form

$$E(v) = E_0 e^{-\frac{1v^2}{2c^2}} \approx e^{-\frac{1}{2}v^2}$$

where $E_0 \approx c \approx 1$ in the graph.

The basic aspect here is that while the relativistic energy will be zero if the speed *v* of the object is equal to the speed of light *c*, the energy in the Gaussian form will be zero if the speed *v* of the object is infinite.

This is another way to suggest a correction to the relativistic expression.

Part 2

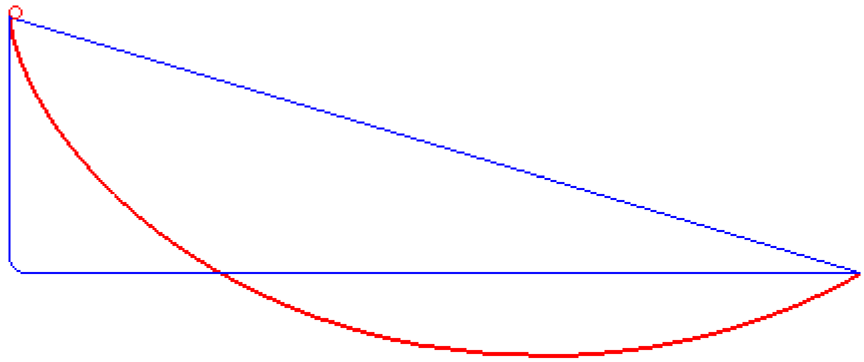
The tautochrone problem

The problem of the brachistochrone, or of the tautochrone (because the brachistochrone is also the tautochrone), is an intriguing one, and important. This is because all objects falling on the brachistochrone from different heights, end up at the bottom of the brachistochrone simultaneously. Additionally if the brachistochrone is the path of least action (or of least time) then this will be the preferred path which all objects, accelerated by gravity, will follow in the universe.

Furthermore, if acceleration, in general, is equivalent to the acceleration of gravity, according to general relativity, then all accelerated objects (no matter what is the cause of the acceleration) will travel on brachistochrones. Therefore the geodesics which objects follow as they travel in spacetime will be brachistochrones.

In this document there will not be a mathematical description and analysis of the equations of the cycloid (the brachistochrone). Instead we will directly focus attention on a condition of synchronism (which I call principle of synchronicity). This condition relates the cycloid curve (the curve of the brachistochrone) to the linear path (the rim of the brachistochrone) which connects the two ends of the curve.

Thus instead of describing the associated distances with respect to the equation of the cycloid, we will treat them as the product of speed by time. Thus, on one hand, there will be the speed v of an object travelling on the brachistochrone at a time T (the time of the brachistochrone, or tautochrone), while, on the other hand, we will have the speed c of a photon travelling on the linear path (the rim of the brachistochrone) at a time equal to (or multiple of) its period. Such a connection will be described with examples later on.



The solution to the brachistochrone problem is not a straight line or some combination thereof, but a cycloid.

Here we may note some historical facts. According to Wikipedia (previous picture), the problem of the brachistochrone was brought forward by Johann Bernoulli in 1696, as follows:

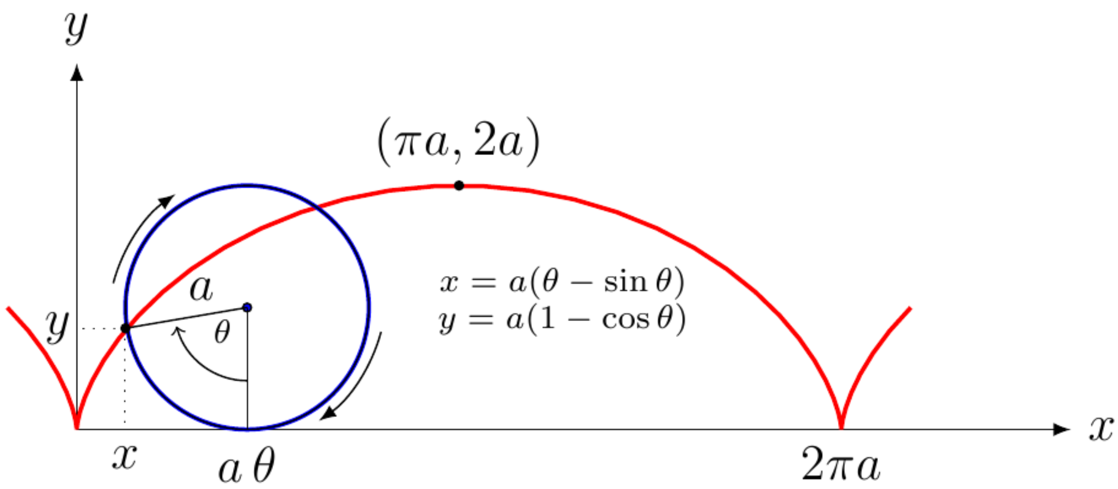
“Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time?”

Bernoulli published his solution the following year, and he went on to show that it yields a cycloid.

Five other mathematicians responded to the problem with solutions: Isaac Newton, Jakob Bernoulli (Johann’s brother), Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus and Guillaume de l’Hôpital.

Earlier, in 1638, Galileo had tried to solve a similar problem, but he drew the conclusion that the arc of a circle is faster than any number of its chords.
[https://en.wikipedia.org/wiki/Brachistochrone_curve]

The following picture illustrates the cycloid:



[https://tex.stackexchange.com/questions/196957/how-can-i-draw-this-cycloid-diagram-with-tikz]

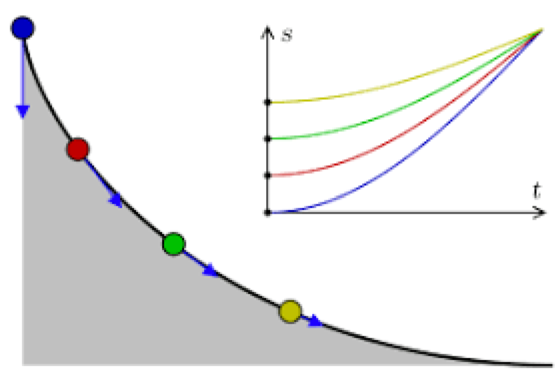
The cycloid equations (in parametric form) are the following,

$x(\theta) = R(\theta - \sin\theta)$

$y(\theta) = R(1 - \cos\theta)$

where *R* is the radius of the rolling (generating) circle which produces the cycloid (in the previous picture the same radius is called *a*).

Significantly enough, the brachistochrone is also the tautochrone:



Four balls slide down a cycloid curve from different positions, but they arrive at the bottom at the same time. The blue arrows show the points' acceleration along the curve. On the top is the time-position diagram.

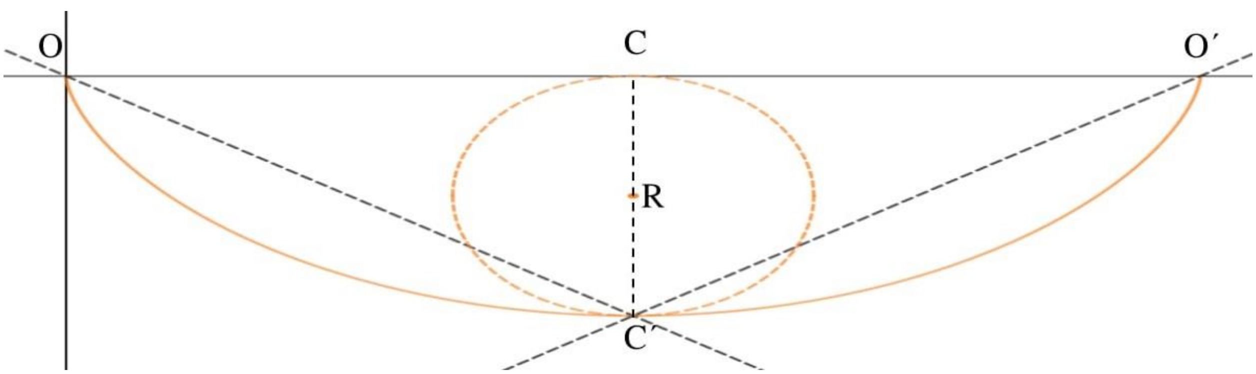
According to Wikipedia (previous picture), a tautochrone or isochrone curve is the curve for which the time taken by an object sliding without friction in uniform gravity to its lowest point is independent of its starting point. The curve is a cycloid, and the time is equal to π times the square root of the radius (of the circle which generates the cycloid) over the acceleration of gravity. The tautochrone curve is the same as the brachistochrone curve for any given starting point.

The tautochrone problem was solved by Christiaan Huygens in 1659. He proved that the time of descent is equal to the time a body takes to fall vertically the same distance as the diameter of the circle that generates the cycloid, multiplied by $\pi/2$. In modern terms, this means that the time of descent is

$$T = \pi \sqrt{\frac{R}{g}}$$

where R is the radius of the circle which generates the cycloid, and g is the gravity of Earth. [https://en.wikipedia.org/wiki/Tautochrone_curve]

Some geometric aspects of the brachistochrone are the following one:



In the previous image, the curved line $S \equiv OC'O'$ is the brachistochrone, generated by the circle of radius $R \equiv CR$, and it is equal to $S=8R$. The linear distance $L \equiv OCO'$ is equal to $L=2\pi R$, while the vertical distance $y \equiv CRC'$ is equal to $y=2R$.

Thus, we have

$$\Delta S = OC'O' = 8R$$
$$\frac{\Delta S}{2} = OC' = O'C' = 4R$$
$$\Delta L = OCO' = 2\pi R$$
$$\Delta x = \frac{\Delta L}{2} = OC = \pi R$$
$$\Delta y = CRC' = 2R$$
$$R = CR = C'R$$

Also comparing the arc $OC' \equiv S/2$ to the chord between the same points OC', which we may call Δr, we take

$$(\Delta r)^2 = (OC')^2 = (OC)^2 + (CC')^2 = (\Delta x)^2 + (\Delta y)^2 = (\pi R)^2 + (2R)^2 = (\pi^2 + 4)R^2 \Rightarrow$$
$$\Delta r = \sqrt{\pi^2 + 4}R,$$
$$\frac{\Delta S/2}{\Delta r} = \frac{4R}{\sqrt{\pi^2 + 4}R} = \frac{4}{\sqrt{\pi^2 + 4}} = \sqrt{\frac{16}{\pi^2 + 4}} = 1.074$$

Furthermore, we have the following ratios,

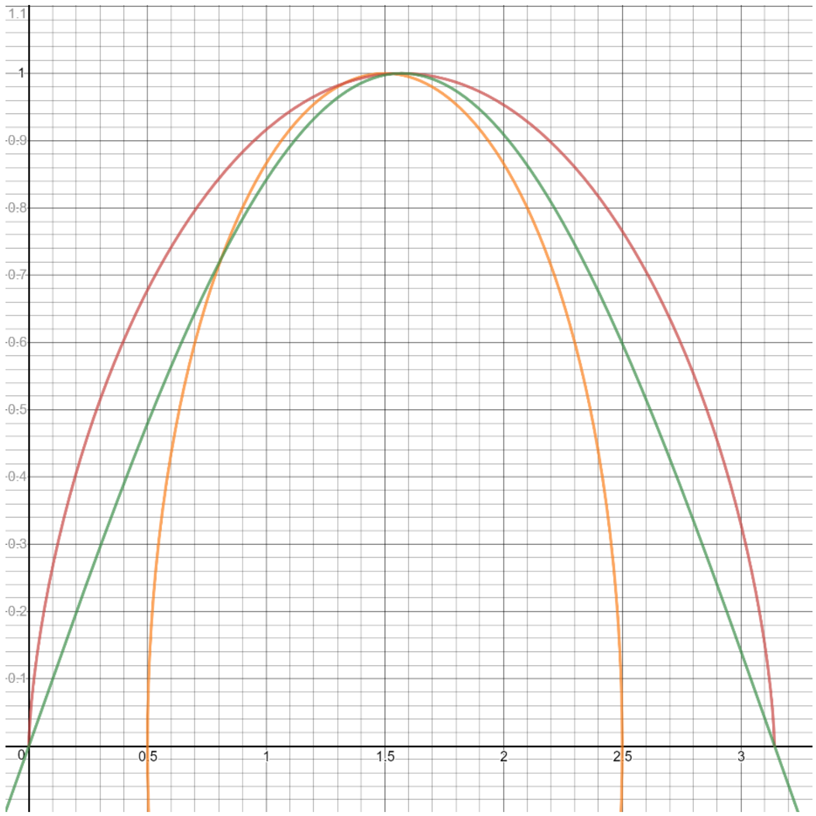
$$\frac{\Delta S/2}{\Delta x} = \frac{4R}{\pi R} = \frac{4}{\pi}$$
$$\frac{\Delta S/2}{\Delta y} = \frac{4R}{2R} = 2$$
$$\frac{\Delta x}{\Delta y} = \frac{\pi R}{2R} = \frac{\pi}{2}$$

Here Δx is the line OC (half the distance ΔL), and $\Delta S/2 \equiv OC'$ is half the arc of the brachistochrone. These are some of the geometric proportions of the problem.

Notes:

For reasons which will become more obvious later on, it will be useful here to compare the brachistochrone to a sinusoidal wave, as well as to the perimeter of a cycle.

This is a related graph:



Red: $[x(t),\ y(t)] = R[(t - \sin t), (1 - \cos t)] = \frac{1}{2}[(t - \sin t), (1 - \cos t)],\ R = \frac{1}{2}$

Green: $y = Rsinx = sinx,\ R = 1$

Orange: $[x(t),\ y(t)] = R[(a + \sin t), (b + \cos t)] = [(1.5 + \sin t), (\cos t)],\ a = 1.5,\ b = 0,$ $R = 1$

In the previous graph, the red line is a brachistochrone (in parametric form), with a generating circle of radius *R=1/2*. The green line corresponds to a sinewave (half its wavelength), with amplitude *R=1*. The orange line (also in parametric form) is half the perimeter of a circle with radius *R=1*.

The exact arc length of the sine curve for a full period is 7.64, thus 3.82 for half a wavelength. [https://en.wikipedia.org/wiki/Sine#Arc_length]

Using the cyclic approximation of a wavelength (that a wavelength λ is equal to the perimeter of a circle whose radius *R* is equal to the amplitude of the wave), the wavelength will be

$\lambda \approx 2\pi R = 2\pi = 6.28, \qquad R = 1 \Rightarrow$

$$\frac{\lambda}{2} = \pi = 3.14$$

On the other hand, the length of the brachistochrone is *8R*, where *R* is the radius of the generating circle. The corresponding brachistochrone in the previous graph has a generating circle of radius *R=1/2*, thus the length *S* of the brachistochrone will be half as much,

$$S = 8R = 8\frac{1}{2} = 4, \qquad R = \frac{1}{2}$$

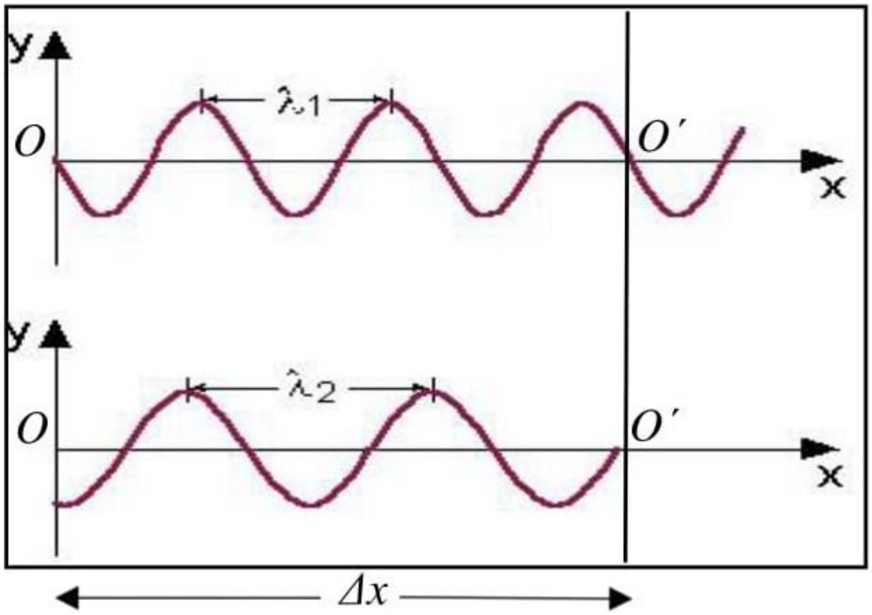
Thus the brachistochrone best approximates a sine wave (being about 5% bigger than the length of a sine wave), rather than the cyclic approximation (where the circle of reference has a perimeter about 15% shorter than the length of the sine wave).

Incidentally the definition and measurement of a ‘wavelength’ is rather ambiguous, as the wavelength itself is an elusive entity. If we take a photograph of a sea wave, we can measure the wavelength as the distance between two peaks, for example. But when the wave is something we cannot see, what we receive is the ‘beat’ per unit of time (the frequency *f*) of the oscillation, or we measure the time it takes for one beat to occur (the period *T*). This is the most realistic and tangible aspect of an invisible wave (such as waves in spacetime). Thus, assuming that the speed of the wave is constant (the speed of light *c* in our case), then the wavelength will just be $\lambda = c/f = cT$. This length represents an oscillation, and also a reference unit of distance (if we use light to measure distances).

The many-photons example

In previous sections we made some preliminary remarks about how the properties (period, wavelength, etc.) of a photon change if we move with respect to the photon. Now we will focus attention on the motion of the photon itself.

This is a related picture:



The many-photons example

Background picture: Shorter wavelength (λ_1) compared to longer wavelength (λ_2)

[https://perg.phys.ksu.edu/vqm/laserweb/Ch-1/F1s1t1p3.htm]

This is a picture of two photons with different wavelengths, λ_1 and λ_2, but the example can be generalized to include many photons with different wavelengths λ_n, where n will be the number of the harmonic.

The problem is the following one:

An observer located at point O uses photons to measure the distance $OO'=\Delta x$. The speed c of the photon is considered (also experimentally verified) fixed, so that if Δt is the time (as recorded on the clock of the observer) it takes the photon to travel the distance from O to O', and back to O, then the observer at point O can determine the distance OO', as

$$\Delta x = c\frac{\Delta t}{2}$$

For simplicity we will remove the ½ factor, assuming that the time Δt refers to the time it takes the photon to travel the distance OO' (without returning to point O'). Still this time interval refers to the clock of the observer. As a consequence the photons (of different wavelengths) do not travel the same distance at the same time, according to their frame of reference. This is because, assuming that the photon is a point traveling on the wave, the path the photon follows is the curved path (on the sinusoidal wave), not the linear path (on the straight line) Δx.

But if the time Δt is fixed (all photons arrive at point O' from point O, thus cover the distance Δx, at the same time Δt), then they *cannot* have the same speed (with respect to their own reference frame on the curved path), since the distances they travel on their curved paths will be different (the smaller the wavelength the longer the distance they travel till they reach point O').

This is a more thorough analysis of the problem. Firstly, the fixed linear distance is

$$\Delta x = c\Delta t$$

Now if λ_n is the wavelength of the n-th photon (assuming many photons of different wavelengths covering the same distance Δx), the distance this photon will have travelled on its curved path across the wave will be

$$L_n = n\lambda_1$$

where n represents the harmonic, or the number of wavelengths λ_n which fit in the distance Δx for the n-th photon. (It may be assumed that also half-wavelengths can fit into the distance, but here we will consider integer wavelengths for simplicity).

The number n of wavelengths will be different for photons of different wavelength λ_n. But we can use a photon of certain wavelength as reference. Thus we will regard a photon of just one wavelength λ_1 (at the first harmonic, $n=1$) as the reference photon, so that the distance this photon travels will approximately be equal to the distance Δx,

$L_1 \equiv \Delta x = n\lambda_1 = \lambda_1, \qquad n = 1$

The rest of the photons will have smaller wavelengths (thus also smaller periods, or bigger frequencies), thus they will travel longer distances,

$L_1 = \lambda_1, \qquad n = 1$

$L_2 = 2\lambda_1, \qquad n = 2$

$L_3 = 3\lambda_1, \qquad n = 3$

...

$L_n = n\lambda_1, \qquad n = n$

Accordingly, the period, which we may call τ_n, of a photon with wavelength λ_n will be n times shorter than the period τ_1 of the reference photon with wavelength λ_1, where by definition,

$$c = \frac{\lambda_n}{\tau_n} = \frac{\lambda_1}{\tau_1},$$

$$n = \frac{\lambda_1}{\lambda_n} = \frac{\tau_1}{\tau_n},$$

$$\tau_n = \frac{1}{n}\tau_1,$$

$$\lambda_n = \frac{1}{n}\lambda_1$$

The time τ_1, which we may also call T_1, can be identified with the time Δt (as we have identified the distance $\lambda_1 \equiv L_1$ with the linear path Δx).

The revealing aspect here is that any photon with wavelength λ_n (n times shorter than the wavelength λ_1 of the photon at the first harmonic, $n=1$), in order to cover the same distance $L_1 \equiv \Delta x$ at the fixed time $T_1 \equiv \Delta t$, will have to travel n times faster than the speed c (i.e. the speed of light), so that if we call v_n the speed of the photon at some harmonic n, it will be:

$$v_n = \frac{L_n}{T_1} \equiv \frac{L_n}{\Delta t} = n\frac{L_1}{T_1} \equiv n\frac{\lambda_1}{\tau_1} = n\frac{c\tau_1}{\tau_1} = nc$$

Still, the speed of light will be constant,

$$c = \frac{\lambda_n}{\tau_n} = \frac{1}{n}\lambda_1 n \frac{1}{\tau_1} = \frac{\lambda_1}{\tau_1} \equiv \frac{L_1}{T_1} \equiv \frac{\Delta x}{\Delta t}$$

Thus for the 'external' observer the speed of any photon will never exceed the speed of light *c*.

As we shall see later on, the speed of light can be strictly reserved for the photons (travelling the linear distance Δx on the rim of the brachistochrone), while the speed *v* will be associated with another particle, connecting the photons, and travelling on the (curved path of) brachistochrone.

Notes:

The time T_1 which we previously mentioned, where,

$$T_1 \equiv \Delta t \equiv \tau_1,$$

$$L_1 \equiv \Delta x \equiv \lambda_1 = cT_1 = c\Delta t = c\tau_1$$

can in fact be identified with the time of the brachistochrone.

To see this, we may define a time T_n, using the formula for the speed v_n,

$$v_n = \frac{L_n}{T_1} = \frac{nL_1}{T_1} = \frac{L_1}{T_n},$$

where

$$T_n = \frac{1}{n}T_1$$

This expression can be justified as follows. Taking the formula for the time of the brachistochrone

$$T = 2\pi\sqrt{\frac{R}{g}} = \sqrt{2\pi\frac{L}{g}}, \qquad L = 2\pi R$$

(where here we wrote 2π, instead of π, considering the total journey, from top to top, on the brachistochrone,)

and setting

$$g_n = n^2 g_1$$

we have that

$$T_1 = \sqrt{2\pi \frac{L_1}{g_1}}$$

$$T_n = \sqrt{2\pi \frac{L_1}{g_n}} = \sqrt{2\pi \frac{L_1}{n^2 g_1}} = \frac{1}{n}\sqrt{2\pi \frac{L_1}{g_1}} = \frac{1}{n} T_1$$

The form of the acceleration g_n will be further explored later on.

Taking here the equivalent expressions for the distance L_1, we have that

$$L_1 = \Delta x = \lambda_1 \Rightarrow$$

$$L_1 = cT_1 = c\Delta t = c\tau_1,$$

$$T_1 = t_1 = \tau_1 \Rightarrow$$

$$L_1 = ncT_n = nc\tau_n \Rightarrow$$

$$L_1 = v_n T_n = v_n \tau_n,$$

$$T_n = \tau_n,$$

$$v_n = nc$$

so that

$$L_1 = c\Delta t = cT_1 \equiv c\tau_1 = v_n T_n \equiv v_n \tau_n \Rightarrow$$

$$v_n T_n = c\tau_1,$$

$$v_n \tau_n = cT_1$$

The previous two formulas represent conditions of the tautochrone (of simultaneity), and relate the time T of (an object travelling on) the brachistochrone to the period τ of the photon.

Conclusively, we may write down the following two basic expressions. On one hand, we have the constant time of the brachistochrone T_1,

$$T_1 = \frac{L_n}{v_n} = \frac{nL_1}{nc} = \frac{L_1}{c} = const.,$$

$$L_n = nL_1, \qquad v_n = nc$$

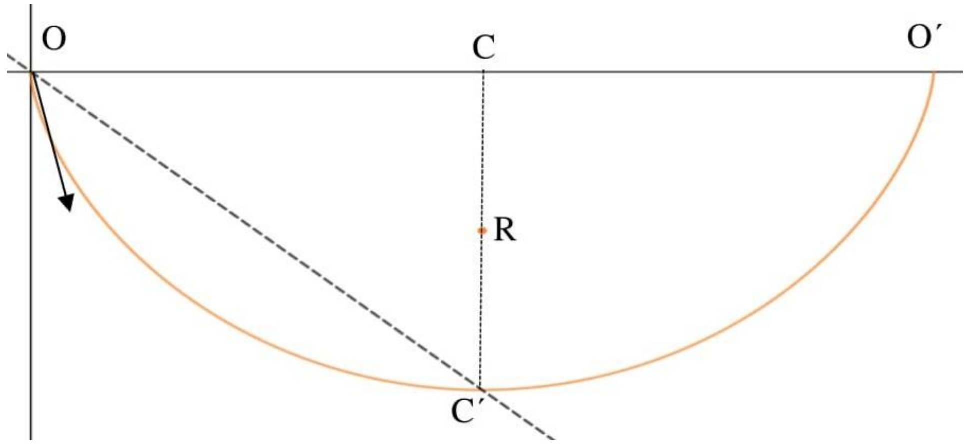
On the other hand, we have the constant speed of light *c*,

$$c = \frac{\lambda_n}{\tau_n} = \frac{\lambda_1}{\tau_1} \equiv \frac{L_1}{T_1} = const.$$

Although these two formulas are interrelated, their origin and implications are fundamentally different. In the later, keeping the speed of the photon constant, both the wavelength and period of the photon (thus the spatial and temporal aspects of spacetime) have to change. In the former, keeping the time of the brachistochrone constant, the speed of the photon has to change, if the distance it travels changes.

Later on this speed will be dissociated from the photon, and will be attributed to- let me call it- the brachiston.

The spaceship in the brachistochrone



This is an example of how we may relate the journey of an object on the brachistochrone to the properties of a photon:

Suppose that a spaceship (black arrow) leaves point *O*, and travels accelerating on the brachistochrone *OC′O′* (orange curve), towards point *C′*. In the beginning of the journey, an observer on the spaceship emits a photon, which travels on the straight line *OC′* (black dotted line). What is the condition that the observer takes back the emitted photon at the end of the journey, at point *C′*?

This is an analysis of the problem:

On one hand, we have the motion of the spaceship. Because the spaceship is accelerating, its path will be that of a brachistochrone, since this is the path of least action (time). Thus it will be the curve *OC′*.

On the other hand, the emitted photon will travel on a straight line, since its speed (the speed of light) is constant, thus the photon does not have any acceleration. If we assume that the photon propagates as a spherical (or plane) wave, then the linear path $\Delta r = OC'$ will be the radius of such a wave. Incidentally, the same distance can also be seen as a vector, whose origin is at point *O*, its end is at point *C′*, and whose measure is $\Delta r = c\Delta t$.

The relationship between the curved and linear path OC' is that between an arc and a chord, respectively, joining the points OC'. These two distances, as we have already seen, are given as follows. Calling ΔS the curved path (the arc) OC', Δr the linear path (the chord) OC', Δx the distance OC, and Δy the distance CC', we have

$$\Delta S = 4R$$

$$\Delta r^2 = \Delta x^2 + \Delta y^2 = (\pi R^2) + (2R)^2 = \pi^2 R^2 + 4R^2 = (\pi^2 + 4)R^2 \Rightarrow$$

$$\Delta r = \sqrt{\pi^2 + 4}R$$

where R is the radius of the brachistochrone, so that

$$\frac{\Delta S}{\Delta r} = \frac{4R}{\sqrt{\pi^2 + 4}R} = \frac{4}{\sqrt{\pi^2 + 4}} = \frac{4}{3.724} = 1.074$$

Thus the two distances are approximately equal to each other, so that we may write

$$\Delta S \approx \Delta r$$

An equivalent way to measure the same distances is to treat them as products of speed by time. In that sense, the distance ΔS will be equal to the speed v of the moving object multiplied by the time T it takes the object to travel the same distance, while the distance Δr will be equal to the speed of light c multiplied by the time Δt it takes light (the photon) to travel that distance. These times need not be the same, so that the speeds need not be the same. But because the two distances ΔS and Δr are approximately equal to each other, we can write,

$$\Delta S = v\Delta T,$$

$$\Delta r = c\Delta t,$$

$$\Delta S \approx \Delta r \Rightarrow$$

$$v\Delta T \approx c\Delta t$$

This is the condition which relates the spatial and temporal properties (v, T) of the moving object to those (c,t) of the photon emitted by the same object.

Now the time T is in fact the time of the brachistochrone. This time is given by the formula

$$\varDelta T = \pi\sqrt{\frac{R}{g}} = \pi\sqrt{\frac{1}{4}\frac{\varDelta S}{g}} = \frac{\pi}{2}\sqrt{\frac{\varDelta S}{g}}, \qquad \varDelta S = 4R$$

where

$$\varDelta S = v\varDelta T = \frac{4}{\pi^2} g(\varDelta T)^2,$$

$$(\varDelta T)^2 = \frac{\pi^2}{4}\frac{\varDelta S}{g}$$

and

$$v = \frac{\varDelta S}{\varDelta T} = \frac{4}{\pi^2} g\varDelta T,$$

$$v^2 = \left(\frac{\varDelta S}{\varDelta T}\right)^2 = \frac{4}{\pi^2} g\varDelta S = \frac{16}{\pi^2} gR$$

On the other hand, the speed of the emitted photon, which is the speed of light *c*, can also be given in relation to the wavelength λ and the period τ of the photon, so that

$$c = \frac{\varDelta\lambda}{\varDelta\tau} = \frac{\varDelta r}{\varDelta t} \Rightarrow$$

$$\varDelta r = c\varDelta t = \frac{\varDelta\lambda}{\varDelta\tau}\varDelta t$$

and

$$\varDelta S = v\varDelta T = \frac{4}{\sqrt{\pi^2 + 4}}\varDelta r \Rightarrow$$

$$v\varDelta T = \frac{4}{\sqrt{\pi^2 + 4}} c\varDelta t = \frac{4}{\sqrt{\pi^2 + 4}}\frac{\varDelta\lambda}{\varDelta\tau}\varDelta t$$

This formula relates the properties of the moving object (its speed *v* and the time *T* it takes for the object to complete the journey), to the properties of the photon emitted by the same object (the wavelength λ, or period τ, of the photon, in relation to the time $\varDelta t$, which can be measured by the clock of an external observer, standing at point *O*).

Here, to simplify things, we will make the following assumptions. First of all the harmonics n of a photon, as we have already seen, are given as

$$n = \frac{\lambda_0}{\lambda_n}$$

where λ_0 is the photon's wavelength before the spaceship begins to move, while λ_n is the photon's wavelength when the spaceship is in motion. Here we have used the index '*0*' (instead of '*1*') for the first state, $n=1$.

The difference in the wavelengths is

$$\Delta\lambda = \lambda_0 - \lambda_n = \lambda_0 - \frac{1}{n}\lambda_0 = n\lambda_n - \lambda_n = \lambda_0\left(1 - \frac{1}{n}\right) = \lambda_n(n-1)$$

If n is sufficiently big, then

$$n \gg 1 \Rightarrow$$
$$\Delta\lambda \approx \lambda_0 = n\lambda_n$$

Furthermore, if the distance Δr is measured in relation to the emitted photon, then if we call N_0 the number of photons (or wavelengths λ_0) which compose the distance Δr in the beginning, and N_n we call the number of photons (or wavelengths λ_n) which compose the distance Δr when the spaceship is moving (at some speed v_n, corresponding to the harmonic n), then we have

$$N_0 = \frac{\Delta r}{\lambda_0}, \qquad N_n = \frac{\Delta r}{\lambda_n}, \qquad n = \frac{N_n}{N_0}$$
$$\Delta r = N_n\lambda_n = N_0\lambda_0, \qquad N_n = nN_0$$

If we suppose that the distance Δr is of size comparable to the initial wavelength λ_0 of the photon, then it will be

$$\Delta r \approx \lambda_0, \qquad n = 1 \Rightarrow$$
$$N_0 \approx 1, \qquad N_n \approx n \Rightarrow$$
$$\Delta r = \lambda_0 = n\lambda_n = c\Delta t = nc\tau_n = c\tau_0,$$
$$\Delta t = \tau_0 = n\tau_n$$

Thus it will be

$$\varDelta S = \frac{4}{\sqrt{\pi^2 + 4}} \varDelta r \approx \varDelta r \Rightarrow$$
$$vT \approx c\varDelta t \approx c\tau_0$$

where the time of the brachistochrone was written as T (instead of $\varDelta T$) for simplicity.

Now if we also suppose that at the first harmonic of the photon, $n=1$, the spaceship travels the distance $\varDelta S$ at the speed of light c, then calling v_0 the speed of the spaceship corresponding to the first harmonic, $n=1$, and v_n its speed corresponding to any harmonic n, and since we have identified the speed v_0 of the spaceship at the first harmonic, $n=1$, with the speed of light c, $v_0=c$, we have

$$\varDelta S \approx \varDelta r \Rightarrow$$
$$vT \approx c\varDelta t \approx c\tau_0,$$
$$v_0 T_0 \approx c\varDelta t \approx c\tau_0, \qquad n = 1, \qquad v_0 = c \Rightarrow$$
$$T_0 \equiv \tau_0 \equiv \varDelta t$$

This way we have identified the time T_0 of the brachistochrone corresponding to the first harmonic, $n=1$, with the period τ_0 of the emitted photon at the same harmonic, $n=1$.

Multiplying the previous equation by the harmonic n, we can also identify the time of the brachistochrone T_n, at any harmonic n, with the period of the emitted photon τ_n, at the same harmonic n,

$$nT_0 = n\tau_0 \Rightarrow$$
$$T_n = \tau_n = \frac{1}{n}\varDelta t = \frac{1}{n}\tau_0 = \frac{1}{n}T_0$$

Thus the speed v_n of the object, corresponding to some harmonic n, will be

$$v_n = nv_0 \equiv nc, \qquad v_0 \equiv c$$

and for the distance to be travelled, it will be

$$\varDelta S \approx \varDelta r = v_n T_n = v_n \tau_n = c\tau_0 = cT_0 = c\varDelta t$$

Form the previous equations, solving for the wavelength λ_n and the period τ_n of the photon,

$$\varDelta\lambda \approx \lambda_0 = n\lambda_n = \frac{v_n}{c}\lambda_n,$$

$$\varDelta\lambda = c\varDelta\tau \approx c\tau_0 = nc\tau_n = v_n\tau_n,$$

$$\lambda_n \ll \lambda_0, \qquad \tau_n \ll \tau_0, \qquad v_n = nc, \qquad n = \frac{\lambda_0}{\lambda_n} \gg 1 \Rightarrow$$

$$\lambda_n = \frac{c}{v_n}\lambda_0 \approx \frac{c}{v_n}\varDelta r$$

$$\tau_n = \frac{c}{v_n}\tau_0 \approx \frac{c}{v_n}\varDelta t$$

we see that what is ‘contracted’ or ‘delayed’ is not the distance to be travelled $\varDelta r$, or the fixed time $\varDelta t$, but the wavelength λ and the period τ of the photon, respectively.

But how is it is possible that the spaceship (moving faster than light) reaches the end of the journey, at point C', before the emitted photon reaches the same point?

If the photon is treated as a ‘point’ which travels all the distance in between, then this is impossible, because causality will be violated. In other words, if the photon is the carrier of the information (thus it also contains the information about the journey of the spaceship) then the spaceship cannot reach point C' before the information of its own arrival reaches the same point.

However, if the photon is treated as a disturbance of spacetime, produced by the oscillations of spacetime, then the emitted photon will be located wherever the spaceship is located at a given time. This is because, since the spaceship travels in spacetime, the observer on the spaceship can measure the oscillations of spacetime (in the form of photons) at the same point. The difference is that the period of the photon will change if the spaceship is in motion, because the period of the photon depends on the speed of the spaceship.

Thus, to answer the question in the original problem, the observer on the spaceship will *always* take back the reference, or ‘emitted,’ photon.

Here is a scheme which shows the difference in the first principles, if we consider spacetime as empty, so that photons are particles which travel in empty space on straight lines, or if we consider spacetime as the medium in which motion occurs, while photons are disturbances of the medium:

On one hand, we have that:

Spacetime is empty.
The observer emits a photon in spacetime.
The photon is a particle which travels on a straight line at the speed of light.
If the observer exceeds the speed of light, he/she will never take back the photon (and causality is violated).

On the other hand, we have that:

Spacetime is an oscillating medium.
The observer can measure the oscillations of spacetime in the form of photons.
The motion of the observer in spacetime disturbs spacetime.
The observer receives the disturbance as a photon with altered period.

As far as the previous condition of simultaneity is concerned,

$$v_n T_n = v_n \tau_n = c\tau_0 = cT_0 = c\Delta t$$

it will be studied in more detail later on in this document.

From the Earth to Alpha Centauri

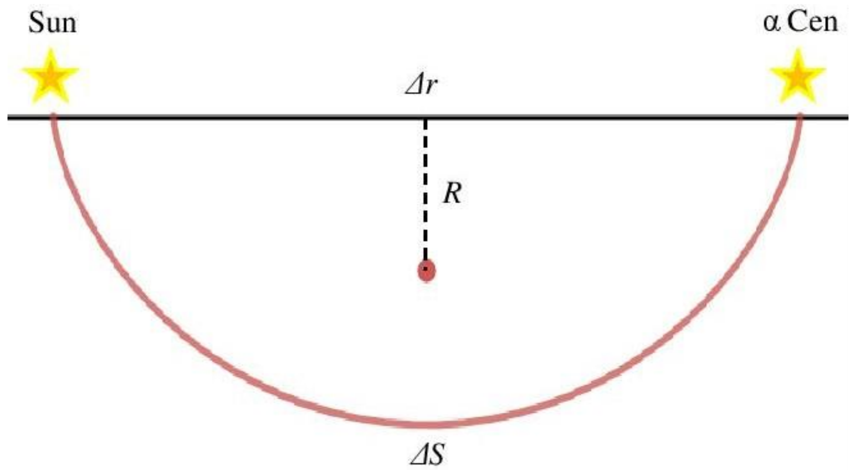


Figure: Journey to Alpha Centauri

Here is an application of the notion of the brachistochrone. For example, a spaceship travels to Alpha Centauri (the nearest star system to the Sun). The distance between the Sun and Alpha Centauri (Δr in the previous image) is estimated at 4.37 light years:

Alpha Centauri is the closest star system to the Solar System, being 4.37 light-years from the Sun. [https://en.wikipedia.org/wiki/Alpha_Centauri]

This is the distance which a photon travels 'on a straight line.' The route of the spaceship, on the other hand, will be a brachistochrone, because the spaceship will have to accelerate to reach a certain speed.

Here we will suppose that the journey involves the total route ΔS from the one end of the brachistochrone (the Sun), to the other end (Alpha Centauri).

Thus if Δr is the linear distance a photon travels, at some time Δt, and at the speed of light c,

$$\Delta r = c\Delta t$$

then the distance ΔS the spaceship travels, at a speed v, and at some time ΔT, will be

$$\Delta S = v\Delta T$$

where ΔT is the time of the brachistochrone

$$\Delta T = 2\pi\sqrt{\frac{R}{g}}$$

and R is the radius of the brachistochrone.

The point here is that we use the photon to estimate the distance, on one hand, and the time of the brachistochrone to estimate when the spaceship arrives at Alpha Centauri, on the other hand.

Now we will suppose that the spaceship has an average acceleration equal to the acceleration of gravity on the surface of the Earth, $g=9.807\ m/s^2$:
The average value (of gravity) at the Earth's surface is, by definition, $9.80665\ m/s^2$.
[https://en.wikipedia.org/wiki/Gravity_of_Earth]

We should note that normally the acceleration g refers to the acceleration of gravity, thus it should refer to the average gravitational field between the Sun and Alpha Centauri. However, due to the equivalence between gravitational charge and inertial mass (thus also the equivalence between gravitational and inertial acceleration), the acceleration g can also be seen as the average acceleration produced by the spaceship's engine.

Incidentally, if we change the units of g from meters per second squared, into light years per year squared, we find out (remarkably enough) that $g \approx 1 ly/y^2$:

$$g = 9.807\frac{m}{s^2} = \left(9.807\frac{m}{s^2}\right)\left(\frac{1}{9.461\times 10^{15}\frac{m}{ly}}\right)\left(3.154\times 10^{7}\frac{s}{y}\right)^2 \Rightarrow$$

$$g = 1.031\frac{ly}{y^2} \approx 1\frac{ly}{y^2}$$

This transformation simplifies the calculation, because, in our case, the distance is in light years.

The radius R of the brachistochrone, with respect to the distance Δr, which we may also call ΔL, is given as

$$\Delta r \equiv \Delta L = 2\pi R = 4.370ly \Rightarrow$$

$$R = \frac{1}{2\pi}\Delta L = \frac{1}{2\pi}4.370ly = 0.6955ly$$

so that, given the acceleration $g=1.031ly/y^2$, the time of the spaceship's arrival will be

$$\Delta T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{0.6955ly}{1.031\frac{ly}{y^2}}} = 2\pi\sqrt{0.6866y^2} = 2\pi(0.8286)y = 5.206y$$

while the spaceship will have travelled a total distance

$$\Delta S = 8R = 8(0.6955ly) = 5.564ly$$

at an average speed equal to

$$v = \frac{\Delta S}{\Delta T} = \frac{5.564ly}{5.206y} = 1.069\frac{ly}{y} = 1.069c$$

If the acceleration doubles, then the time ΔT of the brachistochrone will be $\sqrt{2}$ times less, so that, for the given distance ΔS, the speed v of the spaceship will be $\sqrt{2}$ times as much.

So what happens to the relativistic equations, which imply that nothing can travel faster than light? In fact, we have already seen some formulas which can be reduced to the relativistic formulas if the speed of the moving object is less than the speed of light, while they are they still valid if the speed of the object is greater than light.

The key is that by using the notion of the brachistochrone, the general problem is divided into smaller scales. For example, we have previously supposed that the distance to be traveled can be equated to the wavelength of the reference (the 'emitted') photon. By doing this, we can also identify the speed of the object with the speed of light 'per wavelength.' But if the wavelength is contracted, because of the moving object, then the total speed of the object will be so many times

greater than the speed of light, as many contracted wavelengths compose the total distance. Thus we may say that the relativistic expression will be valid as long as we are confined within one wave of spacetime (a wavelength), while a more general formula will be applied when the object exceeds the speed of light, thus ‘jumps’ to the next wave.

Such ideas, and the corresponding formulas, will become clearer after we introduce the energy of the brachistochrone.

Quantum entanglement

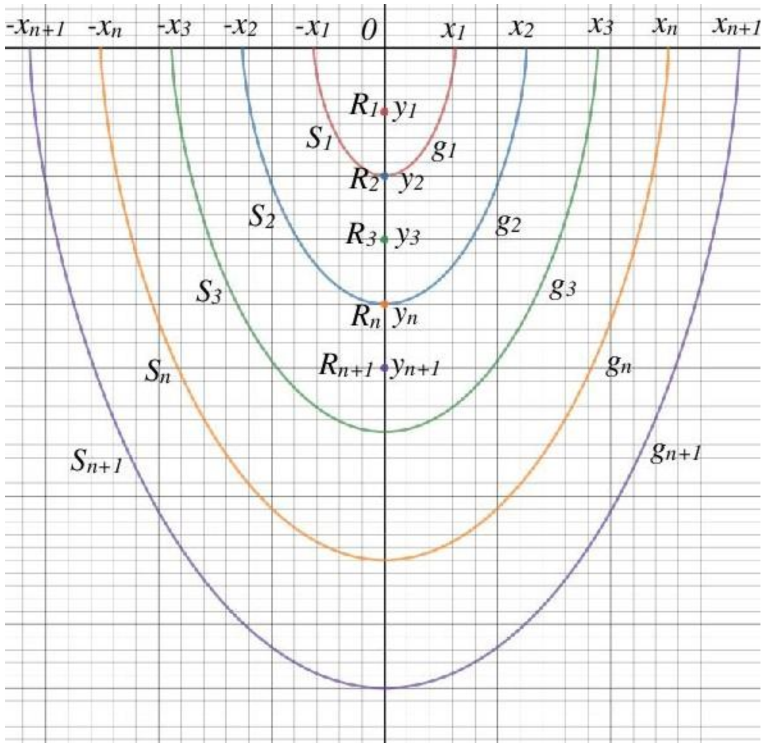


Figure: Two photons are entangled, and travel on the horizontal axis *x* in opposite directions. The entanglement is achieved by a signal (a brachiston) travelling on the curved line (the brachistochrone) which connects the two photons. The time it takes the two photons to interact is the time of the brachistochrone.

In the previous picture, all the curves S_n (brachistochrones) are also tautochrones (brachistochrone= shortest time, tautochrone= same time). This is not the same brachistochrone, on which different objects end up at the bottom at the same time from different heights, but different brachistochrones, on which the same object reaches the bottom of each brachistochrone, supposedly, at different time.

According to experiments on quantum entanglement, the two entangled particles communicate one with the other ‘instantaneously.’ However the time of the brachistochrone may offer the opportunity to measure such ‘instantaneity’ (supposing that the time can be very small, but not zero).

The photons travel on the linear path Δx_n (the interval $[-x_n \dots x_n]$ in the previous graph), so that if it takes for a photon a time Δt_n to travel this distance at the speed of light c, and each interval Δx_n is n times bigger than the initial interval Δx_1, then we have that,

$$\Delta x_1 = c\Delta t_1$$

$$\Delta x_2 = c\Delta t_2 = 2\Delta x_1 = 2c\Delta t_1$$

$$\dots$$

$$\Delta x_n = c\Delta t_n = n\Delta x_1 = nc\Delta t_1$$

where

$$\Delta t_n = n\Delta t_1$$

On the other hand, the brachiston will travel on the curved line S_n, which connects the two ends of the linear distance Δx_n, so that, in general, it will be

$$\Delta S = v\Delta T,$$

where v is the speed of the brachiston, and T is the time of the brachistochrone.

The advantage of having introduced the notion of the brachistochrone, is that each brachistochrone S_n is directly related to the length of the brachistochrone L_n, where the length L_n corresponds to the distance Δx_n by which the two entangled photons are separated at some time interval Δt_n,

$$S = 8R$$

$$L = 2\pi R \Rightarrow$$

$$S = \frac{4}{\pi} L$$

For simplicity we will assume that the length of the brachistochrone S_n is analogous to the length of the brachistochrone's rim L_n, so that

$$\Delta S \approx \Delta L \approx \Delta x \Rightarrow$$

$$L = v\Delta T = c\Delta t$$

The problem here is that we have to define some initial condition, referring to the separation Δx_1 (the interval $[-x_1 \ldots x_1]$ in the previous graph) between the two photons. This can be done if we assume that at the initial separation, $L_1=\Delta x_1=c\Delta t_1$, the brachiston will have to travel as fast as the speed of light, so that the photons are also initially entangled. Thus, it will be

$$L_1 = v_1 T_1$$
$$\Delta x_1 = c\Delta t_1$$
$$v_1 = c \Rightarrow$$
$$L_1 = cT_1 = c\Delta t_1 \Rightarrow$$
$$T_1 \equiv t_1$$

Therefore, due to the initial condition, the time of the brachistochrone T_1, will be equal to the time t_1 of the photon.

As the photons fly apart from each other, the distance L_n will increase,

$$L_1 = \Delta x_1 = v_1 T_1 = cT_1 = c\Delta t_1, \qquad v_1 = c, \qquad T_1 \equiv t_1,$$
$$L_n = \Delta x_n = nL_1 = n\Delta x_1 = nv_1 T_1 = nc\Delta t_1 = c\Delta t_n = v_n T_1 \Rightarrow$$
$$v_n = nc$$

As a consequence, the speed v_n of the brachiston will be n times greater than the speed of the photons c, as the separation L_n increases by a factor of n.

This can be done without violating causality, as the brachiston (although travelling faster than light) will meet the photon (the carrier of information) at the end of the journey.

Another problem is to define some acceleration g, in relation to the time of the brachistochrone T, which by definition is

$$T = 2\pi\sqrt{\frac{R}{g}} = \sqrt{2\pi\frac{L}{g}}, \qquad L = 2\pi R$$

where R is the radius of the brachistochrone.

Solving for the initial separation L_1, we have

$$T_1 = \sqrt{2\pi \frac{L_1}{g_1}} \Rightarrow$$

$$L_1 = cT_1 = \frac{1}{2\pi} g_1 T_1^2 \Rightarrow$$

$$g_1 = 2\pi \frac{c}{T_1} = 2\pi \frac{c^2}{L_1}$$

Defining the speed v_n,

$$v_n = \frac{L_n}{T_1}$$

in such a way that the speed of light c stays constant,

$$c = \frac{L_1}{T_1} = \frac{1}{2\pi} g_1 T_1 = const.$$

we may also solve the time of the brachistochrone T_n for some final separation L_n,

$$T_n = \sqrt{2\pi \frac{L_n}{g_n}} = \sqrt{2\pi \frac{nL_1}{n^2 g_1}} = \sqrt{\frac{1}{n} 2\pi \frac{L_1}{g_1}} = \sqrt{\frac{1}{n}}\sqrt{2\pi \frac{L_1}{g_1}} = \sqrt{\frac{1}{n}} T_1 \Rightarrow$$

$$L_n = v_n T_1 = \frac{1}{2\pi} g_n T_n^2 = \frac{1}{n}\frac{1}{2\pi} g_n T_1^2 \Rightarrow$$

$$g_n = 2\pi \frac{v_n}{T_n^2} T_1 = 2\pi n \frac{v_n}{T_1} = 2\pi n \frac{nc}{T_1} = 2\pi n^2 \frac{c}{T_1} = n^2 2\pi \frac{c}{T_1} = n^2 g_1,$$

where

$$g_n = 2\pi n \frac{v_n}{T_1} = 2\pi nc \frac{v_n}{L_1} = 2\pi \frac{v_n^2}{L_1},$$

$$L_1 = cT_1, \qquad v_n = nc, \qquad c = v_1$$

so that

$$v_n^2 = \frac{1}{2\pi} g_n L_1,$$

$$v_n = \frac{1}{n}\frac{1}{2\pi}\ g_n T_1$$

The formulas above relate different parameters of the system to each other, taking into account the given initial condition.

Incidentally, the same formula can be derived if we consider an energy equation of the form,

$$mg_1R_1 = \frac{1}{2\pi}mg_1L_1 = mv_1^2, \qquad L_1 = 2\pi R_1 \Rightarrow$$

$$v_1^2 = c^2 = \frac{1}{2\pi}g_1L_1, \qquad v_1 = c$$

where the factor of ½ in front of the kinetic energy was dropped for simplicity, and

$$mg_nR_1 = \frac{1}{2\pi}mg_nL_1 = mv_n^2,$$

$$v_n = nv_1 = nc, \qquad v_1 = c \Rightarrow$$

$$g_n = 2\pi\frac{v_n^2}{L_1} = 2\pi n^2\frac{c^2}{L_1} = n^2 2\pi\frac{c^2}{L_1} = n^2 g_1,$$

$$v_n^2 = \frac{1}{2\pi}g_nL_1$$

where

$$L_1 = cT_1 = ct_1, \qquad t_1 = T_1, \qquad n = 1,$$

$$L_n = nL_1 = ncT_1 = nct_1 = ct_n = v_nT_1 = v_n\sqrt{n}T_n = n\sqrt{n}cT_n,$$

$$v = nc, \qquad t_n = nt_1, \qquad T_n = \sqrt{\frac{1}{n}}T_1 = \frac{1}{n\sqrt{n}}t_n$$

Additionally, we have the properties related to the photons, given by the equation

$$c = \frac{\lambda}{\tau} \Rightarrow$$

$$c = \frac{\lambda_1}{\tau_1} = \frac{\lambda_n}{\tau_n}$$

where λ is the photon's wavelength, and τ is its period.

If, in addition to the initial condition that the initial speed v_1 of the brachiston which entangles the two photons is the speed of light c,

$$L_1 = v_1 T_1 \equiv c T_1 = c t_1, \qquad T_1 \equiv t_1, \qquad v_1 \equiv c$$

we also assume that the initial separation L_1 of the photons is comparable to their wavelength λ_1, then it will be

$$L_1 \approx \lambda_1 \Rightarrow$$

$$L_1 = c T_1 = c t_1 = c \tau_1, \qquad T_1 = t_1 = \tau_1$$

so that, for some final separation L_n, we have that

$$L_n = n L_1 = n c T_1 \equiv n c t_1 \equiv n c \tau_1 = n^2 c \tau_n = c t_n = v_n T_1 = v_n \sqrt{n} T_n = n \sqrt{n} c T_n,$$

where

$$T_1 \equiv t_1 \equiv \tau_1,$$

$$n = \frac{\lambda_0}{\lambda_n} = \frac{\tau_0}{\tau_n}, \qquad n = [1,2,3, \dots],$$

and

$$T_n = \sqrt{\frac{1}{n}} T_1 = \frac{1}{n\sqrt{n}} t_n = \sqrt{n} \tau_n,$$

$$v_n \tau_n = n c \frac{1}{n} \tau_1 = c \tau_1 \equiv c T_1$$

The last equation represents a condition of simultaneity, or synchronicity.

The aspect that the photons' period τ_n decreases, while their harmonic n increases, is a consequence of the definitions and assumptions given above.

This can also be seen from the last equation, where the speed of the brachiston v_n is inversely proportional to the period of the photon τ_n, since their product is constant.

A more general relationship between the same two parameters, v_n and τ_n, can be taken if we assume an energy equation of the form,

$$E_n = \frac{hc}{\lambda_n^2} L_1 = n^2 \frac{hc}{\lambda_1^2} L_1 = n^2 m v_1^2 = m v_n^2, \qquad v_n = n v_1,$$

$$E_1 = \frac{hc}{\lambda_1^2} L_1 = m v_1^2 = mc^2, \qquad v_1 = c,$$

$$\frac{hc}{\lambda_n^2} L_1 = m v_n^2 \Rightarrow$$

$$v_n^2 = \frac{hc}{m} \frac{L_1}{\lambda_n^2} = \frac{h}{mc\tau_n^2} L_1 = \frac{h}{mc\tau_n^2} cT_1 = \frac{h}{m\tau_n^2} T_1$$

$$v_1 = c, \qquad L_1 = cT_1$$

The last equation shows that the period of the photon τ_n, is inversely proportional to the speed of the brachiston v_n.

If here we set

$$m = \frac{h}{c\lambda_1} = \frac{h}{c^2\tau_1}$$

equating thus, for some reason, the mass *m* of the brachiston, to the mass-equivalent of a region of spacetime of size λ_1, we take

$$v_n^2 = \frac{h}{m\tau_n^2} T_1 = \frac{c^2\tau_1}{\tau_n^2} T_1 \equiv c^2 \frac{T_1^2}{\tau_n^2}, \qquad T_1 \equiv \tau_1 \Rightarrow$$

$$v_n \tau_n = cT_1$$

so that we retrieve the condition of synchronicity we met earlier.

Notes:

The main point is that we cannot define the time of the brachistochrone, thus the time of the entanglement, independently of the photons, that is their wavelength, or period.

This can be seen if we add another energy term of the form Mc^2, and using the indices '*B*' for '*Brachistochrone*,' and '*p*' for '*photon*,' so that

$$E_B = \frac{1}{2\pi} mgL_B = M_B c^2 = \frac{hc}{\lambda_p^2} L_B$$

where the mass M_B refers to the mass of the brachistochrone (the mass in the region of spacetime where the object of mass m travels).

Replacing

$$M_B = \frac{h}{c\lambda_p^2} L_B$$

and identifying the acceleration g with the common gravitational acceleration,

$$g = \frac{GM_B}{R_B^2} = 4\pi^2 \frac{GM_B}{L_B^2}, \qquad L_B = 2\pi R_B$$

we have

$$g = 4\pi^2 \frac{GM_B}{L_B^2} = 4\pi^2 \frac{Gh}{c\lambda_p^2 L_B}$$

Thus the time of the brachistochrone T_B will be

$$T_B = \sqrt{2\pi \frac{L_B}{g}} = \sqrt{\frac{1}{2\pi} \frac{c\lambda_p^2}{Gh} L_B^2} = \sqrt{2\pi \frac{c\lambda_p^2}{Gh} R_B^2} = \sqrt{\frac{c\lambda_p^2}{\hbar G} R_B^2} = \sqrt{\frac{c}{\hbar G}} \lambda_p R_B = \sqrt{\frac{c^3}{\hbar G}} \tau_p R_B,$$

$$\lambda_p = c\tau_p, \qquad \hbar = \frac{h}{2\pi}$$

where λ_p and τ_p is respectively the wavelength and period of the reference (the entangled) photons, and $\hbar$ is (the reduced) Planck constant.

If in the last relationship we substitute

$$t_P = \sqrt{\frac{\hbar G}{c^5}}$$

where t_P is Planck time, we take

$$T_B = \frac{\tau_p}{t_P}\frac{R_B}{c}$$

This is a simple formula to estimate the time it takes for the entanglement to occur.

This time however cannot be defined independently of the period of the entangled photons. Thus the problem is how one can measure such period, since, presumably, it will be extremely low (even smaller than Planck time), if the time of the entanglement T_B is miniscule.

As far as the aforementioned energy relationships are concerned, they will become clearer after the energy of the brachistochrone is introduced.

Black holes

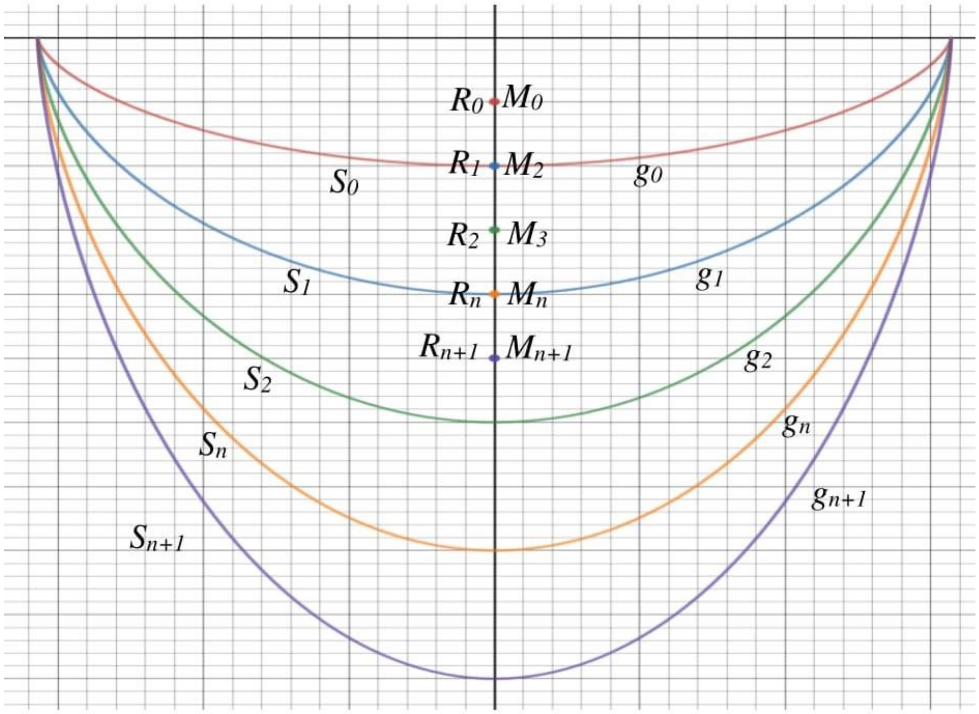


Image: Illustration of a black hole as a gravitational well of brachistochrones

Another application of the brachistochrone, besides quantum entanglement (although here it is at the macroscopic level), is black holes.

The previous graph is similar to the next to last graph, describing quantum entanglement, with one exception. Here the edges of the successive brachistochrones S_n are repositioned at the edges of the first brachistochrone S_0, corresponding to the points $(-x_1,x_1)$ in the next to last graph).

Although such brachistochrones (with the exception of the first one S_0) are not 'true' brachistochrones (their equations do not correspond to that of a cycloid), each one respectively has the same total length L_n (thus also the same radius R_n) with the 'true' brachistochrones in the next to last graph.

This way, even if the shape of the brachistochrone is 'deformed,' we can treat the deformed curves as new brachistochrones S_n at a higher harmonic n. Thus the last graph may illustrate the black hole as a 'gravitational well' of brachistochrones.

Beyond this, the considerations made in the previous section, about quantum entanglement, including the related formulas, can also be applied in the case of black holes.

An additional element is that the mass M_B (or M_n at some state n) is assumed to be concentrated on the focus (the center of the generating circle) of the respective brachistochrone S_n. Thus the mass M_n represents the mass of the black hole or radius R_n.

As far as far the acceleration g_n is concerned, we should mention that, in the original problem of the brachistochrone, this acceleration is the acceleration of gravity. However, because of the equivalence between the gravitational charge and the inertial mass of an object (that an accelerating object creates in spacetime a field equivalent to a gravitational field), we may treat the gravitational acceleration as equivalent to the acceleration generated, or gained, by the moving object, during its journey on the brachistochrone.

Therefore, if M_n is the mass of the black hole, and R_n is its radius, at some state n, then for the acceleration g_n we may write that,

$$g_0 = \frac{c^2}{R_0} = \frac{GM_0}{R_0^2}, \qquad n = 1$$

$$g_n = \frac{v_n^2}{R_0} = \frac{GM_n}{R_0^2} = n^2 g_0$$

$$M_n = n^2 M_0, \qquad v_n = nc$$

where here we have used the index '*0*' for the state $n=1$.

The proposal that the mass of the black hole M_B increases by a factor of n^2, as the state n increases, can be based on the following general energy equation,

$$E_B = \frac{1}{2\pi} mgL_B = M_B c^2 = mv^2 = \frac{hc}{\lambda^2} L_B$$

which can also be expressed in relation to the state n,

$$E_n = \frac{1}{2\pi} m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2} L_0,$$

$$E_0 = \frac{1}{2\pi} m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{\lambda_0^2} L_0, \qquad n = 1$$

$$n = \frac{\lambda_0}{\lambda_n} \Rightarrow$$

$$E_n = \frac{hc}{\lambda_n^2} L_0 = n^2 \frac{hc}{\lambda_0^2} L_0 = n^2 E_0,$$

so that

$$M_n c^2 = n^2 M_0 c^2 \Rightarrow$$

$$M_n = n^2 M_0$$

The energy equation will be explored in more detail in the next section.

The significance of this result is that the speed of an object moving across the black hole will be,

$$M_B c^2 = m_0 v_0^2 \Rightarrow$$

$$v_0^2 = \frac{M_B}{m_0} c^2$$

Since, presumably, the inertial mass m_0 of an object will be much less than the mass M_B of the black hole, then the final speed v_0 of the object will be much greater than the speed of light c.

Such a result is possible only if we distinguish between the two masses m_0 and M_B, where the latter mass refers to the properties of spacetime (the black hole in this case), while the former mass refers to the properties of an object (moving across the same region).

The time it takes for the object to move across the black hole, will be the time T_B of the brachistochrone,

$$T_B = 2\pi\sqrt{\frac{R_B}{g}} = \sqrt{2\pi\frac{L_B}{g}}$$

$$g = \frac{GM_B}{R_B^2} = 4\pi^2\frac{GM_B}{L_B^2}$$

$$T_B = 2\pi\sqrt{\frac{R_B^3}{GM_B}}$$

Using now the equations for the energy and for the gravitational acceleration, we can take an expression for the radius R_B of the black hole (presumably the radius of its event horizon),

$$E_B = \frac{1}{2\pi}m_0 g L_B = m_0 g R_B = M_B c^2 = \frac{1}{2}m_0 v^2, \qquad L_B = 2\pi R_B$$

$$g = \frac{v^2}{R_B} = \frac{GM_B}{R_B^2} \Rightarrow$$

$$\frac{Gm_0 M_B}{R_B} = M_B c^2 = \frac{1}{2}m_0 v^2 \Rightarrow$$

$$R_B = \frac{2GM_B}{v^2} = \frac{Gm_0}{c^2}$$

Here we may compare the previous formula to that which gives the Schwarzschild radius of a black hole,

$$R_S = \frac{2GM}{c^2}$$

We see that, while Schwarzschild formula does not depend on the speed of the object, the formula for R_B tells us that the event horizon of the black hole shrinks if the object moves faster than light.

Furthermore, the expression

$$R_B = \frac{Gm_0}{c^2}$$

is also interesting in the sense that it reveals a relationship between the inertial mass m_0 of the object, and the radius R_B of the black hole.

The dependence of the properties of the black hole, to those of a material object moving with respect to the black hole, can also be seen if we use a photon of reference, as we have already done in quantum entanglement. If λ is the photon's wavelength, then, we have

$$M_B c^2 = \frac{1}{2} m v^2 = \frac{hc}{\lambda^2} L_B \Rightarrow$$

$$\lambda^2 = \frac{h}{M_B c} L_B = \frac{2hc}{m v^2} L_B$$

Among other things, the previous formula shows the dependence of the wavelength of the reference photon on the speed of the object.

The reference photons can either be seen as photons emitted by the moving object, or as fluctuations on the event horizon L_B (the rim) of the black hole of radius R_B, perceived in the form of photons.

Incidentally, we should mention here that in the formula for the wavelength λ,

$$\lambda^2 = \frac{h}{M_B c} L_B$$

$$g = 2\pi \frac{v^2}{L_B} = 4\pi^2 \frac{G M_B}{L_B^2} \Rightarrow$$

$$M_B = \frac{1}{2\pi} \frac{L_B v^2}{G} \Rightarrow$$

$$\lambda^2 = 2\pi \frac{hG}{c} \frac{1}{v^2}$$

if we equate the speed of the objet v to the speed of light c, we take

$$v = c \Rightarrow$$

$$\lambda^2 = 2\pi \frac{hG}{c^3} \equiv l_P^2,$$

$$l_P = 2\pi r_P, \qquad r_P = \sqrt{\frac{\hbar G}{c^3}}, \qquad \hbar = \frac{h}{2\pi}$$

where l_P stands for Planck length, and r_P stands for Planck radius.

The last relationship implies that an observer on board an object approaching, or crossing, the event horizon of a black hole at the speed of light, $v=c$, will measure the wavelength λ of the oscillating event horizon equal to Planck length l_P. If his/her speed is greater than the speed of light, the same wavelength will be even shorter than Planck length, as the black hole, according to him/her, will shrink to a singularity of zero dimension, if his/her speed becomes infinite.

Notes:

Here is a way to estimate the wavelength λ decrease (or frequency f increase) of the reference photon, with respect to the speed v of an object of mass m. From the energy equation we have,

$$E_B = \frac{hc}{\lambda^2} L_B = \frac{1}{2} m v^2$$

Noting λ_0 and λ' the initial and final wavelength respectively (or f_0 and f' the frequency), and E_0 and E' the corresponding energy, we take

$$E_0 = \frac{hc}{\lambda_0^2} L_0 = \frac{hf_0^2}{c} L_0,$$

$$E' = \frac{hc}{\lambda'^2} L_0 = \frac{hf'^2}{c} L_0,$$

where

$$c = f'\lambda' = f_0\lambda_0 = const.$$

For the total energy we have

$$E_0 = E' + \frac{1}{2} m v^2 \Rightarrow$$

$$\frac{hf_0^2}{c} L_0 = \frac{hf'^2}{c} L_0 + \frac{1}{2} m v^2 \Rightarrow$$

$$\frac{1}{2} m v^2 = \frac{hf_0^2}{c} L_0 - \frac{hf'^2}{c} L_0 = \frac{h}{c} L_0 (f_0^2 - f'^2)$$

Thus, for the speed v of the object we take

$$v^2 = \frac{2h}{mc} L_0 (f_0^2 - f'^2) \Rightarrow$$

$$v = \sqrt{\frac{2h}{mc} L_0 (f_0^2 - f'^2)}$$

and for the final frequency f' of the reference photon we take

$$\frac{hf'^2}{c} L_0 = \frac{hf_0^2}{c} L_0 - \frac{1}{2} m v^2 \Rightarrow$$

$$f'^2 = \frac{c}{hL_0} \left(\frac{hf_0^2}{c} L_0 - \frac{1}{2} m v^2 \right) = f_0^2 - \frac{mc}{2hL_0} v^2 \Rightarrow$$

$$f' = \sqrt{f_0^2 - \frac{mc}{2hL_0} v^2} = \sqrt{f_0^2 - f_m^2},$$

where

$$f_m = \sqrt{\frac{mc}{2hL_0} v^2}$$

The quantity f_m has units of frequency, and it can be associated with the motion of the object.

If we set

$$v \approx 0, \qquad f_m \approx 0 \Rightarrow$$

$$f' \approx f_0$$

then the object will be motionless, and there will not be any change in the frequency f'.

On the other hand, setting

$$f' \approx 0 \Rightarrow$$

$$f_0 \approx f_m = \sqrt{\frac{mc}{2hL_0} v_0^2},$$

$$v_0 = \sqrt{\frac{2h}{mc} L_0 f_0}$$

then the frequency f_m of the moving object will be equal to the initial frequency f_0 of the photon, and the frequency f' of the fluctuations of spacetime will seize.

The point is that since the reference photons are fluctuations (oscillations) of the medium (spacetime), and because the energy of spacetime is proportional to the frequency of those fluctuations, as the moving object transforms this energy into its own kinetic energy, then the frequency of the fluctuations will decrease, as long as the speed of the object increases.

This is analogous to the common experience we have when moving in the sea with a ship. If the speed of the ship increases, we perceive that the frequency of the waves increases (we hear the waves hitting the ship more often). But this is the frequency related to our motion. In contrast, an external observer may expect that the frequency of the waves decreases, although such a change is miniscule compared to the vast amount of energy stored in the waves.

The frequency f' corresponds to the reduced frequency of the oscillations within a harmonic n. A simpler formula can be taken assuming the motion at any harmonic n, as follows.

Solving the previous formula which gives the frequency f', for the initial frequency f_0,

$$f' = \sqrt{f_0^2 - \frac{mc}{2hL_0} v^2} = \sqrt{f_0^2 - f_m^2}, \qquad f_m = \sqrt{\frac{mc}{2hL_0} v^2} \Rightarrow$$

$$f_0 = \sqrt{f'^2 + \frac{mc}{2hL_0} v^2} = \sqrt{f'^2 + f_m^2}$$

and supposing that the speed of the object is large enough, so that

$$f_m \gg f' \Rightarrow$$

$$f_0 \approx f_m = \sqrt{\frac{mc}{2hL_0} v_0^2}$$

then we may identify the frequency f_0 with the frequency of the oscillations at the first state, $n=1$, so that, with respect to the harmonics n of the fluctuations, the speed v_n and the frequency f_n, at any state n, will be

$$f_n = \sqrt{\frac{mc}{2hL_0} v_n^2}$$

This can also be seen from the energy E_n, at any state n,

$$E_n = \frac{hf_n^2}{c} L_0 = M_n c^2 = \frac{1}{2} m v_n^2 \Rightarrow$$

$$v_n = \sqrt{\frac{2M_n}{m}} c = \sqrt{\frac{2hL_0}{mc}} f_n,$$

$$f_n = \sqrt{\frac{M_n c^3}{hL_0}} = \sqrt{\frac{mc}{2hL_0}} v_n$$

Setting now

$$v_n = n v_0 = nc, \qquad v_0 = c, \qquad n = 1$$

we take

$$f_n = \sqrt{\frac{mc}{2hL_0}} v_n = \sqrt{\frac{mc}{2hL_0}} nc = n \sqrt{\frac{mc^3}{2hL_0}} = n f_0,$$

$$f_0 = \sqrt{\frac{mc^3}{2hL_0}} = \sqrt{\frac{M_0 c^3}{hL_0}} = \sqrt{\frac{1}{2\pi} \frac{c^5}{Gh}}, \qquad M_0 = \frac{R_0 c^2}{G} = \frac{1}{2\pi} \frac{L_0 c^2}{G}$$

The last frequency, which, incidentally, will be equal to Planck frequency (in analogy to the wavelength λ_0 we earlier saw),

$$\lambda_0 = \sqrt{2\pi \frac{hG}{c^3}} \equiv l_P, \qquad v_0 = c$$

$$f_0 = \frac{c}{\lambda_0} = c\sqrt{\frac{1}{2\pi}\frac{c^3}{hG}} = \sqrt{\frac{1}{2\pi}\frac{c^5}{hG}} \equiv f_P$$

can be seen as the frequency of the oscillations on the event horizon of the black hole, as the object of mass *m* crosses the event horizon at the speed of light.

In any case, the notion of the brachistochrone can be used to represent a two- dimensional model of a black hole, where the successive excited states *n* of the black hole depend on the speed of an object approaching or crossing the black hole at a corresponding speed v_n, so that, according to an observer on this object, both the radius R_n of the event horizon of the black hole, and the wavelength λ_n of its oscillations, will shrink.

Energy in the brachistochrone

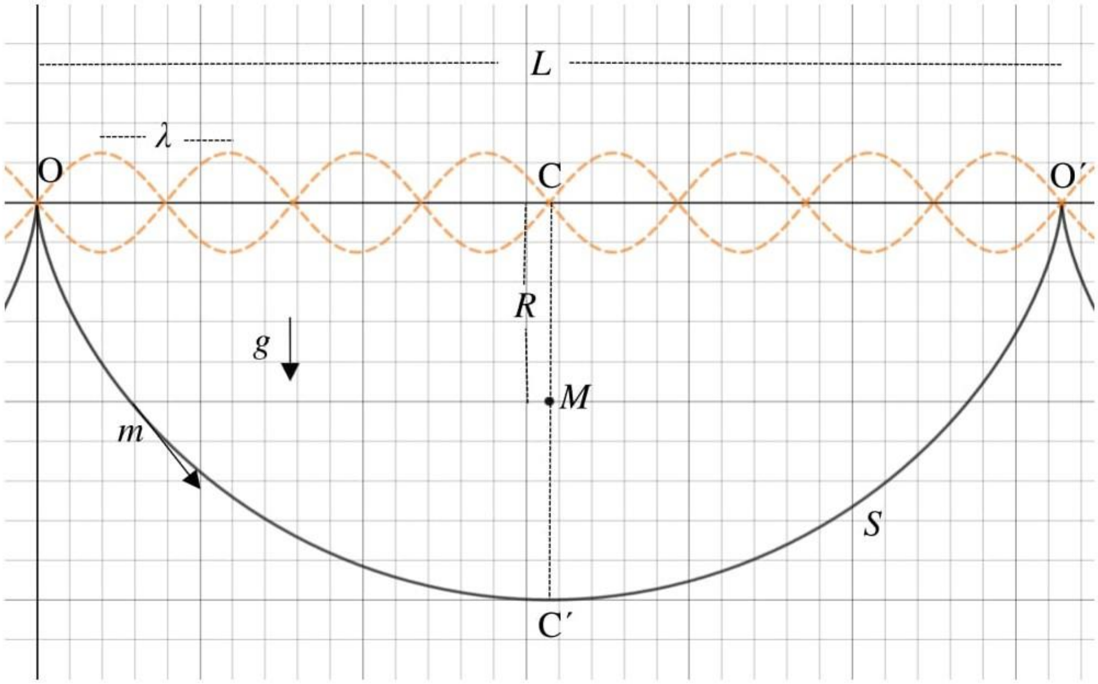


Image: Representation of a brachistochrone S of radius R and of mass M, together with a reference photon of wavelength λ. The mass of the brachistochrone is seen as concentrated on the focus (black dot), and produces a uniform acceleration g.

In this picture, the wavelength λ of the reference photon is half a wavelength. The photon is represented as a standing wave. We may also note that the wavelength of the photon can be seen as a small brachistochrone of size $\lambda/2$, within the larger brachistochrone S.

Furthermore, the mass m represents an object travelling on the brachistochrone S. The mass M is the mass of the brachistochrone, and it is considered to be concentrated on the focus of the brachistochrone (at point M). The acceleration g is supposed to be uniform, covering the whole area of the brachistochrone.

This is a first approach to the energy of the brachistochrone, the successive terms of which we have already seen in the previous couple of sections,

$$E_B = \frac{1}{2\pi} m_0 g L_B = M_B c^2 = m_0 v^2 = \frac{hc}{\lambda^2} L_B$$

$$E_B \equiv E_n, \qquad L_B \equiv L_0, \qquad M_B \equiv M_n, \qquad m \equiv m_0, \qquad g \equiv g_n, \qquad v \equiv v_n, \qquad \lambda \equiv \lambda_n$$
$$n = \frac{\lambda_0}{\lambda} \equiv \frac{\lambda_0}{\lambda_n}$$

In that form, the above quantities represent total energies, at some state of the brachistochrone n, although the state is not explicitly noted here.

The first term can also be written as,
$$\frac{1}{2\pi} m_0 g L_B = m_0 g R_B = m_0 \frac{GM_B}{R_B},$$

where
$$g = \frac{GM_B}{R_B^2} = 4\pi^2 \frac{GM_B}{L_B^2}, \qquad L_B = 2\pi R_B,$$

and it is the common gravitational energy.

The second term is Einstein energy of the brachistochrone ($E=mc^2$). But here the distinction was made between the mass of the object m_0, and the mass of the brachistochrone M_B.

The third term is the common kinetic energy of an object of mass m_0, if its final speed if v. We can assume that the total kinetic energy mv^2 is twice the average kinetic energy ($1/2mv^2$), so that we drop the ½ factor.

The fourth term refers to the Planck energy of a photon of wavelength λ,
$$\varepsilon = \frac{hc}{\lambda}$$

so that the total Planck energy stored in the distance L_B will be
$$E_B = \frac{L_B}{\lambda} \frac{hc}{\lambda} = \frac{hc}{\lambda^2} L_B$$

where the quantity

$$N = \frac{L_B}{\lambda}$$

will be the number of photons of wavelength λ which compose the distance L_B (at some state of the brachistochrone).

Here is a comparison between some of the pairs,

$$M_B c^2 = m_0 v^2 \Rightarrow$$

$$\frac{M_B}{m_0} = \frac{v^2}{c^2}$$

This is an indication that the speed v of an object moving on the brachistochrone can exceed the speed c of light, if the inertial mass m_0 of the moving object is smaller than the mass of the brachistochrone M_B,

$$\frac{M_B}{m_0} = \frac{v^2}{c^2}$$

$$M_B \gg m_0 \Rightarrow$$

$$v \gg c$$

Expressions for the masses m_0 and M_B can also be taken from the following pairs,

$$\frac{1}{2\pi} m_0 g L_B = 2\pi m_0 \frac{GM_B}{L_B} = M_B c^2 = m_0 v^2, \qquad g = \frac{GM_B}{R_B} = 4\pi^2 \frac{GM_B}{L_B}, \qquad L_B = 2\pi R_B \Rightarrow$$

$$m_0 = \frac{1}{2\pi} \frac{c^2}{G} L_B = \frac{c^2 R_B}{G},$$

$$M_B = \frac{1}{2\pi} \frac{v^2}{G} L_B = \frac{v^2 R_B}{G}$$

From these expressions we take back the previous ratio,

$$\frac{M_B}{m_0} = \frac{v^2}{c^2}$$

From the same energy pairs, we may also take expressions for the speed *v* of the moving object, and for the speed of light *c*,

$$\frac{1}{2\pi}m_0 g L_B = m_0 g R_B = M_B c^2 = m_0 v^2 \Rightarrow$$

$$v^2 = \frac{1}{2\pi} g L_B = g R_B$$

$$c^2 = \frac{1}{2\pi}\frac{m_0}{M_B} g L_B = \frac{m_0}{M_B} g R_B$$

Thus a formula of the form

$$c^2 = gR$$

will be a special case, if the inertial mass of the object moving on the brachistochrone is comparable to the mass of the brachistochrone, $m_0 \approx M_B$.

The energy E_B can also be expressed with respect to the time T_B of the brachistochrone,

$$E_B = \frac{1}{2\pi}m_0 g L_B = 2\pi m_0 \frac{GM_B}{L_B} = M_B c^2 = m_0 v^2 = \frac{hc}{\lambda^2} L_B \Rightarrow$$

$$E_B = \frac{1}{4\pi^2} m_0 g^2 T_B^2 = 4\pi^2 m_0 \frac{GM_B}{g T_B^2} = M_B c^2 = m_0 v^2 = \frac{1}{2\pi}\frac{hg}{c}\frac{T_B^2}{\tau^2} = \frac{h}{c\tau^2} v T_B$$

where

$$g = \frac{GM_B}{R_B^2} = 4\pi^2 \frac{GM_B}{L_B^2},$$

$$T_B = 2\pi\sqrt{\frac{R_B}{g}} = \sqrt{2\pi\frac{L_B}{g}} \Rightarrow$$

$$L_B = 2\pi R_B = \frac{1}{2\pi} g T_B^2$$

$$v^2 = \frac{1}{2\pi} g L_B = g R_B$$

$$v = \frac{L_B}{T_B} = \frac{1}{2\pi} g T_B$$

$$c^2 = \frac{1}{2\pi}\frac{m_0}{M_B}gL_B = \frac{m_0}{M_B}gR_B$$

$$c = \frac{\lambda}{\tau}$$

Solving for the time of the brachistochrone T_B, we take,

$$M_B c^2 = \frac{1}{2\pi}\frac{hg}{c}\frac{T_B^2}{\tau^2} = 2\pi\frac{hGM_B}{cL_B^2}\frac{T_B^2}{\tau^2}, \qquad g = 4\pi^2\frac{GM_B}{L_B^2} \Rightarrow$$

$$T_B^2 = \frac{1}{2\pi}\frac{c^3}{hG}L_B^2\tau^2 = 2\pi\frac{c^3}{hG}R_B^2\tau^2 = \frac{c^3}{\hbar G}R_B^2\tau^2, \qquad L_B = 2\pi R_B, \qquad \hbar = \frac{h}{2\pi} \Rightarrow$$

$$T_B = \sqrt{\frac{c^3}{\hbar G}}R_B\tau = \frac{R_B}{r_P}\tau, \qquad r_P = \sqrt{\frac{\hbar G}{c^3}}$$

where r_P is Planck radius.

Also, for the period τ of the reference photon, we have that

$$m_0 v^2 = \frac{h}{c\tau^2}vT_B \Rightarrow$$

$$m_0 v = \frac{h}{c\tau^2}T_B \Rightarrow$$

$$\tau^2 = \frac{h}{c}\frac{T_B}{m_0 v} \Rightarrow$$

$$\tau = \sqrt{\frac{h}{c}\frac{T_B}{m_0 v}}$$

The last formula shows the dependence of the period τ of the reference photon on the speed v of the moving object.

Notes:

The next to last formula, referring to the time T_B of the brachistochrone, has the advantage of including all the fundamental constants: the speed of light c of relativity, Planck constant h of quantum mechanics, and Newton's constant G of gravity.

The same formula can also be solved for the radius of the brachistochrone R_B,

$$T_B = \sqrt{\frac{c^3}{\hbar G}} R_B \tau = \frac{R_B}{r_P} \tau \Rightarrow$$

$$R_B = \sqrt{\frac{\hbar G}{c^3}} \frac{T_B}{\tau} = r_P \frac{T_B}{\tau}, \qquad r_P = \sqrt{\frac{\hbar G}{c^3}}$$

If in this formula we equate the two times, the time of the brachistochrone T_B, and the period of the reference photon τ, we take the formula for Planck length l_P (here Planck radius r_P),

$$T_B \equiv \tau \Rightarrow$$

$$R_B = \sqrt{\frac{\hbar G}{c^3}} \equiv r_P,$$

On the other hand, if we equate the period of the reference photon τ to Planck time t_P,

$$l_P = 2\pi r_P = c t_P,$$

$$t_P = 2\pi \sqrt{\frac{\hbar G}{c^5}}$$

(where a factor of 2π was added here for reasons of compatibility),

we take

$$\tau \equiv t_P \Rightarrow$$

$$R_B = \sqrt{\frac{\hbar G}{c^3}} \frac{T_B}{\tau} \equiv \sqrt{\frac{\hbar G}{c^3}} \frac{T_B}{t_P} = r_P \frac{T_B}{t_P} = \frac{1}{2\pi} c T_B$$

And because, more generally, as we have already seen,

$$R_B = \frac{1}{2\pi} v T_B$$

we have also to conclude that if the period τ of the reference photon is equal to Planck time t_P, then the speed v of the moving object will be the speed of light c.

Consequently if the speed v of the objet exceeds the speed of light c, then the period τ of the reference photon will be smaller than Planck time t_P. If this is true, then the energy equation of the brachistochrone, in addition to bringing together the aspects of the macrocosm with those of the microcosm, it also expands the limits.

However if we don't have a way to distinguish between the different parameters, such as the lengths L_B, λ and l_P, or the corresponding times T_B, τ, and t_P, or the different masses M_B and m_0, which appear in the system of the brachistochrone, we won't be able to make the necessary reductions, or to find for the same quantities a physical meaning.

Energy of the harmonics *n*

The energy equation of the brachistochrone we saw in the previous section

$$E_B = \frac{1}{2\pi} m_0 g L_B = M_B c^2 = m_0 v^2 = \frac{hc}{\lambda^2} L_B$$

does not distinguish between the different harmonics *n* of the reference photon. But if assume that the wavelength λ of the reference photon in the previous equation refers to some harmonic *n*, then, if λ_0 is the wavelength of the photon at the first harmonic, $n=1$, and λ_n is its wavelength at some harmonic *n*, we have that

$$n = \frac{\lambda_0}{\lambda} \equiv \frac{\lambda_0}{\lambda_n} \Rightarrow$$

$$E_n = \frac{hc}{\lambda_n^2} L_0 = n^2 \frac{hc}{\lambda_0^2} L_0,$$

$$E_0 = \frac{hc}{\lambda_0^2} L_0, \qquad n = 1$$

where the length $L_0 \equiv L_B$ of the brachistochrone, thus also its radius $R_0 \equiv R_B$, are constants.

Thus the energy equation, if it expresses the harmonic *n*, can be written as,

$$E_0 = m_0 g_0 R_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{\lambda_0^2} L_0, \qquad n = 1$$

$$E_n = m_0 g_n R_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2} L_0 = n^2 \frac{hc}{\lambda_0^2} L_0 = n^2 E_0,$$

$$n = \frac{\lambda_0}{\lambda_n}$$

A key element is that the energy E_n, at some harmonic *n*, will be n^2 times greater than the energy E_0, at the first harmonic, $n=1$.

Because of this fact, we may take the following relationships,

$$g_0 = \frac{v_0^2}{R_0} = \frac{GM_0}{R_0^2},$$

$$g_n = n^2 g_0 = n^2 \frac{v_0^2}{R_0} = n^2 \frac{GM_0}{R_0^2} \Rightarrow$$

$$g_n = \frac{v_n^2}{R_0} = \frac{GM_n}{R_0^2},$$

$$v_n = nv_0,$$

$$v_n^2 = g_n R_0,$$

$$M_n = n^2 M_0$$

Also, if we define the time T_n of the brachistochrone, at some corresponding state n, with respect to the initial time T_0,

$$T_0 = 2\pi\sqrt{\frac{R_0}{g_0}} = \sqrt{2\pi\frac{L_0}{g_0}}, \qquad L_0 = 2\pi R_0$$

then it will be

$$T_n = \sqrt{2\pi\frac{L_0}{g_n}} = \sqrt{2\pi\frac{L_0}{n^2 g_0}} = \frac{1}{n}\sqrt{2\pi\frac{L_0}{g_0}} = \frac{1}{n}T_0,$$

where

$$L_0 = \frac{1}{2\pi}g_0 T_0^2 = \frac{1}{2\pi}g_n T_n^2 = const.$$

Subsequently, defining the speed v as the ratio

$$v = \frac{L}{T}$$

it will also be

$$v_0 = \frac{L_0}{T_0} = \frac{1}{2\pi}g_0 T_0, \qquad L_0 = \frac{1}{2\pi}g_0 T_0^2$$

$$v_0^2 = \frac{1}{2\pi}g_0 L_0$$

$$v_n = nv_0 = \frac{L_0}{T_n} = \frac{1}{2\pi}g_n T_n, \qquad T_n = \frac{1}{n}T_0, \qquad L_0 = \frac{1}{2\pi}g_n T_n^2$$

$$v_n^2 = \frac{1}{2\pi} g_n L_0$$

If here we set $v_0=c$, we have the following formulas,

$$c = \frac{L_0}{T_0} = \frac{1}{2\pi} g_0 T_0$$

$$c^2 = \frac{1}{2\pi} g_0 L_0$$

However more generally, from the energy equation we have that

$$M_n c^2 = m_0 v_n^2,$$

$$M_0 c^2 = m_0 v_0^2, \qquad n = 1 \Rightarrow$$

$$\frac{v_n}{c} = \sqrt{\frac{M_n}{m_0}},$$

$$\frac{v_0}{c} = \sqrt{\frac{M_0}{m_0}}, \qquad n = 1$$

so that the speed of the object v_0 at the first harmonic, $n=1$, will be equal to the speed of light, only if its mass m_0 is equal to the mass M_0 of the brachistochrone.

$$v_0 \equiv c \Leftrightarrow M_0 \equiv m_0$$

Comparing the two speeds, we have

$$\frac{v_0}{c} = \frac{L_0}{T_0}\frac{\tau_0}{\lambda_0} = \sqrt{\frac{M_0}{m_0}}$$

$$\frac{v_n}{c} = n\frac{v_0}{c} = n\frac{L_0}{T_0}\frac{\tau_0}{\lambda_0} = \frac{L_0}{T_n}\frac{\tau_0}{\lambda_0} = \frac{L_0}{T_n}\frac{\tau_n}{\lambda_n} = \sqrt{\frac{M_n}{m_0}}, \qquad T_n = \frac{1}{n}T_0 = \frac{\tau_n}{\tau_0}T_0$$

so that it is expected that the speed v_n of the object can exceed of light c, as it moves at higher harmonics.

Such relationships and their consequences will be better understood, after we distinguish between the harmonics and the states of the brachistochrone, in what follows.

Energy states

Up till now we have used the terms 'harmonic' (referring to the harmonic n of the reference photon), and 'state' (referring to the brachistochrone) indiscriminately. For example, in the previous section, it was supposed that the number of photons which fit in the length of the brachistochrone L_0, at some harmonic n, is equal to that number n. However this is not necessarily true. More generally, if N_0 is the number of photons of wavelength λ_0 which fit in the length L_0 of the brachistochrone at the first harmonic, $n=1$, and N_n is the number of photons of wavelength λ_n which fit in the length L_0 of the brachistochrone at some harmonic, n, then it will be

$$L_0 = N_0\lambda_0, \qquad N_0 = \frac{L_0}{\lambda_0}, \qquad n = 1,$$

$$L_0 = N_n\lambda_n, \qquad N_n = \frac{L_0}{\lambda_n} = n\frac{L_0}{\lambda_0} = nN_0,$$

$$n = \frac{\lambda_0}{\lambda_n} = \frac{N_n}{N_0},$$

$$L_0 = N_0\lambda_0 = N_n\lambda_n$$

This helps us distinguish between the energy E_n, at some state n, and the energy per wavelength λ_n, which we may call ε_n, at the same state,

$$E = \frac{1}{2\pi}mgL = Mc^2 = mv^2 = \frac{hc}{\lambda^2}L \rightarrow$$

$$E_0 = \frac{1}{2\pi}m_0g_0L_0 = M_0c^2 = m_0v_0^2 = \frac{hc}{\lambda_0^2}L_0 = N_0\frac{hc}{\lambda_0},$$

$$N_0 = \frac{L_0}{\lambda_0}, \qquad n = 1,$$

$$E_n = n^2E_0 = \frac{1}{2\pi}m_0g_nL_0 = M_nc^2 = m_0v_n^2 = \frac{hc}{\lambda_n^2}L_0 = N_n\frac{hc}{\lambda_n} = nN_n\frac{hc}{\lambda_0} = n^2N_0\frac{hc}{\lambda_0} = n^2\frac{hc}{\lambda_0^2}L_0,$$

$$N_n = \frac{L_0}{\lambda_n}, \qquad n = \frac{\lambda_0}{\lambda_n} = \frac{N_n}{N_0}$$

Defining now an energy ε_0 per photon of wavelength λ_0, then for the first harmonic, or state, $n=1$, it will be

$$\varepsilon_0 = \frac{hc}{\lambda_0} = \mu_0 c^2,$$

and the total energy at the first state can also be written as,

$$E_0 = N_0\varepsilon_0 = N_0\frac{hc}{\lambda_0} = \frac{L_0}{\lambda_0}\frac{hc}{\lambda_0} = \frac{hc}{\lambda_0^2}L_0 = N_0\mu_0 c^2, \qquad N_0 = \frac{L_0}{\lambda_0}, \qquad n = 1$$

Similarly, for any state n, the energy ε_n per photon of wavelength λ_n will be

$$\varepsilon_n = \frac{hc}{\lambda_n} = n\frac{hc}{\lambda_0} = n\varepsilon_0 = n\mu_0 c^2 = \mu_n c^2, \qquad \mu_n = n\mu_0, \qquad n = \frac{\lambda_0}{\lambda_n},$$

so that the total energy at the same state n will be,

$$E_n = n^2 E_0 = \frac{hc}{\lambda_n^2}L_0 = N_n\frac{hc}{\lambda_n} = nN_n\frac{hc}{\lambda_0} = n^2 N_0\frac{hc}{\lambda_0} = n^2\frac{hc}{\lambda_0^2}L_0 = N_n\mu_n c^2,$$

$$N_n = \frac{L_0}{\lambda_n}, \qquad n = \frac{\lambda_0}{\lambda_n} = \frac{N_n}{N_0}$$

Therefore the energy equations can be written in the following general form,

$$E_0 = \frac{1}{2\pi}m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{\lambda_0^2}L_0 = N_0\mu_0 c^2,$$

$$E_n = n^2 E_0 = \frac{1}{2\pi}m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2}L_0 = n^2\frac{hc}{\lambda_0^2}L_0 = N_n\mu_n c^2,$$

$$N_0 = \frac{L_0}{\lambda_0}, \qquad N_n = \frac{L_0}{L_n}, \qquad n = \frac{\lambda_0}{\lambda_n} = \frac{N_n}{N_0},$$

$$\varepsilon_0 = \frac{1}{N_0}E_0 = \frac{\lambda_0}{L_0}E_0 \Rightarrow$$

$$\varepsilon_0 = \frac{\lambda_0}{L_0}\frac{1}{2\pi}m_0 g_0 L_0 = \frac{\lambda_0}{L_0}M_0 c^2 = \frac{\lambda_0}{L_0}m_0 v_0^2 = \frac{\lambda_0}{L_0}\frac{hc}{\lambda_0^2}L_0 \Rightarrow$$

$$\varepsilon_0 = \frac{1}{2\pi}m_0 g_0 \lambda_0 = \frac{\lambda_0}{L_0}M_0 c^2 = \frac{\lambda_0}{L_0}m_0 v_0^2 = \frac{hc}{\lambda_0} = \mu_0 c^2$$

$$\varepsilon_n = n\varepsilon_0 \Rightarrow$$

$$\varepsilon_n = n\frac{1}{2\pi}m_0 g_0 \lambda_0 = n\frac{\lambda_0}{L_0}M_0 c^2 = n\frac{\lambda_0}{L_0}m_0 v_0^2 = n\frac{hc}{\lambda_0} = n\mu_0 c^2 \Rightarrow$$

$$\varepsilon_n = n\frac{1}{2\pi}m_0 \frac{1}{n^2}g_n n\lambda_n = n\frac{n\lambda_n}{L_0}\frac{1}{n^2}M_n c^2 = n\frac{n\lambda_n}{L_0}m_0\frac{1}{n^2}v_n^2 = n\frac{hc}{n\lambda_n} = n\frac{1}{n}\mu_n c^2 \Rightarrow$$

$$\varepsilon_n = \frac{1}{2\pi}m_0 g_n \lambda_n = \frac{\lambda_n}{L_0}M_n c^2 = \frac{\lambda_n}{L_0}m_0 v_n^2 = \frac{hc}{\lambda_n} = \mu_n c^2,$$

$$g_n = n^2 g_0, \qquad \lambda_0 = n\lambda_n, \qquad M_n = n^2 M_0, \qquad v_n = nv_0, \qquad \mu_n = n\mu_0$$

Here we should mention that the definitions,

$$\varepsilon_0 = \frac{hc}{\lambda_0} = \mu_0 c^2, \qquad \mu_0 = \frac{hc}{\lambda_0}$$

$$\varepsilon_n = \frac{hc}{\lambda_n} = \mu_n c^2, \qquad \mu_n = \frac{hc}{\lambda_n}$$

bring us back to the principle of wave-particle duality, which we earlier saw in this document, in the form of the following equations,

$$p_m = mv, \qquad E_m = mv^2$$

$$p_\mu = \mu c, \qquad E_\mu = \mu c^2$$

where m is the mass of a material object, and μ is the mass of the wave 'accompanying' that object.

But here the notion of the brachistochrone of mass M_n, (at some state n), and the mass of the brachistochrone μ_n per wavelength λ_n of the reference photon, makes such a principle much clearer: The wave of mass μ_n which accompanies the object of mass m_0 is associated with the wavelength λ_n of the reference photon (thus the mass μ_n refers to the mass of the wave, and not to the inertial mass m_0 of the object), while the same mass μ_n, as a sub-division of the total mass M_n of the brachistochrone, is not the mass of a photon.

Now a similar division of the brachistochrone can be made with respect to Planck length l_P, instead of the wavelength λ of the reference photon, if Planck length l_P is used as a reference unit of length.

Thus in addition to the formulas

$$n = \frac{\lambda_0}{\lambda_n}, \qquad N_0 = \frac{L_0}{\lambda_0}, \qquad N_n = \frac{L_0}{\lambda_n}$$

which give the number of photons N_n which fit in the length of the brachistochrone L_0 at some state or harmonic n, we may suppose that the distance L_0 is composed of a number N_P of Planck lengths l_P. Since the distance L_0 is supposed to be fixed, the number N_P will be constant, independently of the state n, since Planck length is fixed. Thus, we have

$$L_0 = N_0\lambda_0 = N_n\lambda_n = N_P l_P \Rightarrow$$

$$N_P = \frac{L_0}{l_P} = N_0\frac{\lambda_0}{l_P} = N_n\frac{\lambda_n}{l_P},$$

$$n = \frac{\lambda_0}{\lambda_n}, \qquad N_0 = \frac{L_0}{\lambda_0}, \qquad N_n = \frac{L_0}{\lambda_n}$$

This way we can add another term in the energy equation, referring to the number N_P, as follows. At the first state, $n=1$, it will be

$$\varepsilon_0 = \frac{hc}{\lambda_0} = \mu_0 c^2,$$

$$E_0 = N_0\varepsilon_0 = \frac{L_0}{\lambda_0}\varepsilon_0 = \frac{L_0}{\lambda_0}\mu_0 c^2 = \frac{hc}{\lambda_0^2}L_0, \qquad N_0 = \frac{L_0}{\lambda_0},$$

$$E_0 = \frac{hc}{\lambda_0^2}L_0 = \frac{hc}{\lambda_0^2}\frac{l_P}{l_P}L_0 = \frac{hc^2}{c\lambda_0^2}\frac{l_P}{l_P}L_0 = \frac{h}{cl_P}c^2\frac{l_P}{\lambda_0^2}L_0 = m_P c^2\frac{l_P}{\lambda_0^2}L_0 = m_P c^2\frac{l_P}{\lambda_0^2}N_P l_P$$

$$= m_P c^2\frac{l_P^2}{\lambda_0^2}N_P,$$

$$m_P = \frac{h}{cl_P}, \qquad L_0 = N_P l_P \Rightarrow$$

$$E_0 = N_0\mu_0 c^2 = \frac{L_0}{\lambda_0}\mu_0 c^2 = \frac{hc}{\lambda_0^2}L_0 = \frac{l_P}{\lambda_0^2}L_0 m_P c^2 = \frac{l_P^2}{\lambda_0^2}N_P m_P c^2$$

where m_P is Planck mass.

Thus, at any state n, it will be

$$\varepsilon_n = \frac{hc}{\lambda_n} = n\frac{hc}{\lambda_0} = n\mu_0 c^2 = \mu_n c^2, \qquad n = \frac{\lambda_0}{\lambda_n}$$

$$E_n = n^2 E_0 = \frac{hc}{\lambda_n^2} L_0 = n^2 \frac{hc}{\lambda_0^2} L_0 = N_n \mu_n c^2,$$

$$E_n = n^2 E_0 = n^2 \frac{l_P}{\lambda_0^2} L_0 m_P c^2 = \frac{\lambda_0^2}{\lambda_n^2} \frac{l_P}{\lambda_0^2} L_0 m_P c^2 = \frac{l_P}{\lambda_n^2} L_0 m_P c^2 = \frac{l_P^2}{\lambda_n^2} N_P m_P c^2,$$

$$L_0 = N_0 \lambda_0 = N_n \lambda_n = N_P l_P$$

This way we have a term in the energy equation referring to Planck length.

Before writing down the full energy equation, including all terms, we will make a third step, in order to include in the energy equation a term which will refer to the graviton mass, which we may call m_g. As we previously did with the numbers N and N_P, referring to the number of photons, or Planck lengths, respectively, which fit in the distance L_0, we may introduce a number N', which will refer to the number of gravitons of mass m_g which compose the mass M_n of the brachistochrone, at some state n.

Thus if N_0' is the number of gravitons which compose the mass M_0 of the brachistochrone at the first state,

$$n = 1 \Rightarrow$$

$$E_n \equiv E_0$$

$$M_0 = N_0' m_g, \qquad N_0' = \frac{M_0}{m_g}$$

then, if N_n' is the number of gravitons which compose the mass M_n of the brachistochrone at some state n, it will be

$$n = n \Rightarrow$$

$$E_n = n^2 E_0,$$

$$M_n c^2 = n^2 M_0 c^2,$$

$$M_n = n^2 M_0 \Rightarrow$$

$$M_n = N_n' m_g = n^2 M_0 = n^2 N_0' m_g, \qquad N_n' = \frac{M_n}{m_g} = n^2 \frac{M_0}{m_g} = n^2 N_0', \qquad n = \frac{\lambda_0}{\lambda_n}$$

This way we have the following additional energy term, with respect to the number N' of gravitons of mass m_g,

$$E_0 = N_0\mu_0 c^2 = \frac{L_0}{\lambda_0}\mu_0 c^2 = \frac{L_0}{\lambda_0}\frac{h}{c\lambda_0}c^2 = \frac{L_0}{\lambda_0^2}\lambda_g\frac{h}{c\lambda_g}c^2 = \frac{L_0}{\lambda_0^2}\lambda_g m_g c^2 = N_0{}'m_g c^2 = M_0 c^2$$

$$E_n = n^2 E_0 = n^2\frac{L_0}{\lambda_0^2}\lambda_g m_g c^2 = \frac{L_0}{\lambda_n^2}\lambda_g m_g c^2 = n^2 N_0{}'m_g c^2 = N_n{}'m_g c^2 = n^2 M_0 c^2 = M_n c^2$$

where λ_g is the graviton's wavelength.

Gathering now together all the terms, we have for the numbers n, N, N_P, and N', that

$$n = \frac{\lambda_0}{\lambda_n},$$

$$N_0 = \frac{L_0}{\lambda_0}, \qquad N_n = \frac{L_0}{\lambda_n},$$

$$N_P = \frac{L_0}{l_P},$$

$$N_0{}' = \frac{M_0}{m_g} = \frac{L_0}{\lambda_0^2}\lambda_g, \qquad N_n{}' = \frac{M_n}{m_g} = \frac{L_0}{\lambda_n^2}\lambda_g,$$

$$L_0 = N_0\lambda_0 = N_n\lambda_n = N_P l_P$$

$$M_n = n^2 M_0 = n^2 N_0{}'m_g = N_n{}'m_g$$

so that the energy E_0 at the first state will be

$$E_0 = N_0\mu_0 c^2 = \frac{l_P^2}{\lambda_0^2}N_P m_P c^2 = N_0{}'m_g c^2 \Rightarrow$$

$$E_0 = \frac{L_0}{\lambda_0}\mu_0 c^2 = \frac{l_P^2}{\lambda_0^2}\frac{L_0}{l_P}m_P c^2 = \frac{L_0}{\lambda_0^2}\lambda_g m_g c^2, \qquad N_0{}' = \frac{L_0}{\lambda_0^2}\lambda_g \Rightarrow$$

$$E_0 = \frac{L_0}{\lambda_0}\mu_0 c^2 = \frac{L_0}{\lambda_0^2}l_P m_P c^2 = \frac{L_0}{\lambda_0^2}\lambda_g m_g c^2,$$

$$E_0 = \frac{1}{2\pi}m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{\lambda_0^2}L_0 = \frac{L_0}{\lambda_0}\mu_0 c^2 = \frac{L_0}{\lambda_0^2}l_P m_P c^2 = \frac{L_0}{\lambda_0^2}\lambda_g m_g c^2$$

and the energy E_n, at any state n, will be

$$E_n = n^2 E_0 \Rightarrow$$

$$E_n = n^2 \frac{1}{2\pi} m_0 g_0 L_0 = n^2 M_0 c^2 = n^2 m_0 v_0^2 = n^2 \frac{hc}{\lambda_0^2} L_0 = n^2 \frac{L_0}{\lambda_0} \mu_0 c^2 = n^2 \frac{L_0}{\lambda_0^2} l_P m_P c^2$$

$$= n^2 \frac{L_0}{\lambda_0^2} \lambda_g m_g c^2 \Rightarrow$$

$$E_n = \frac{1}{2\pi} m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{\lambda_0^2}{\lambda_n^2} \frac{L_0}{\lambda_0} \mu_0 c^2 = \frac{\lambda_0^2}{\lambda_n^2} \frac{L_0}{\lambda_0^2} l_P m_P c^2$$

$$= \frac{\lambda_0^2}{\lambda_n^2} \frac{L_0}{\lambda_0^2} \lambda_g m_g c^2 \Rightarrow$$

$$E_n = \frac{1}{2\pi} m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{L_0}{\lambda_n^2} \lambda_0 \mu_0 c^2 = \frac{L_0}{\lambda_n^2} l_P m_P c^2 = \frac{L_0}{\lambda_n^2} \lambda_g m_g c^2$$

while the corresponding energy per wavelength will be,

$$\varepsilon_0 = \frac{1}{N_0} E_0 = \frac{\lambda_0}{L_0} E_0 \Rightarrow$$

$$\varepsilon_0 = \frac{1}{2\pi} m_0 g_0 \lambda_0 = \frac{\lambda_0}{L_0} M_0 c^2 = \frac{\lambda_0}{L_0} m_0 v_0^2 = \frac{hc}{\lambda_0} = \mu_0 c^2 = \frac{\lambda_P}{\lambda_0} m_P c^2 = \frac{\lambda_g}{\lambda_0} m_g c^2$$

$$\varepsilon_n = n\varepsilon_0 = \frac{1}{2\pi} m_0 g_n \lambda_n = \frac{\lambda_n}{L_0} M_n c^2 = \frac{\lambda_n}{L_0} m_0 v_n^2 = \frac{hc}{\lambda_n} = \mu_n c^2 = \frac{\lambda_P}{\lambda_n} m_P c^2 = \frac{\lambda_g}{\lambda_n} m_g c^2$$

Here are some relationships which we can derive from the energy equation of the brachistochrone,

$$E_0 = \frac{1}{2\pi} m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{\lambda_0^2} L_0 = \frac{L_0}{\lambda_0} \mu_0 c^2 = \frac{L_0}{\lambda_0^2} l_P m_P c^2 = \frac{L_0}{\lambda_0^2} \lambda_g m_g c^2$$

$$E_n = \frac{1}{2\pi} m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{\lambda_0}{\lambda_n^2} L_0 \mu_0 c^2 = \frac{l_P}{\lambda_n^2} L_0 m_P c^2 = \frac{\lambda_g}{\lambda_n^2} L_0 m_g c^2,$$

with respect to the masses m_0, μ_n, and M_n,

$$M_0 = m_0 \frac{v_0^2}{c^2} = \frac{L_0}{\lambda_0} \mu_0 = \frac{L_0}{\lambda_0^2} l_P m_P = \frac{L_0}{\lambda_0^2} \lambda_g m_g,$$

$$M_n = n^2 M_0 = m_0 \frac{v_n^2}{c^2} = \frac{L_0}{\lambda_n} \mu_n = \frac{L_0}{\lambda_n^2} l_P m_P = \frac{L_0}{\lambda_n^2} \lambda_g m_g,$$

$$\mu_n \lambda_n = \mu_0 \lambda_0 = m_P \lambda_P = m_g \lambda_g$$

or with respect to the speeds v_n and c,

$$m_0 v_0^2 = M_0 c^2,$$
$$m_0 v_n^2 = M_n c^2 \Rightarrow$$
$$v_0^2 = \frac{M_0}{m_0} c^2 = \frac{L_0}{\lambda_0} \frac{\mu_0}{m_0} c^2 = \frac{l_P}{\lambda_0^2} L_0 \frac{m_P}{m_0} c^2 = \frac{\lambda_g}{\lambda_0^2} L_0 \frac{m_g}{m_0} c^2,$$
$$v_n^2 = n^2 v_0^2 = \frac{M_n}{m_0} c^2 = \frac{L_0}{\lambda_n} \frac{\mu_n}{m_0} c^2 = \frac{l_P}{\lambda_n^2} L_0 \frac{m_P}{m_0} c^2 = \frac{\lambda_g}{\lambda_n^2} L_0 \frac{m_g}{m_0} c^2,$$
$$c^2 = \frac{m_0}{M_0} v_0^2 = \frac{m_0}{M_n} v_n^2$$

Thus we see, among other things, that the speed v_n of the object and the speed of light c can be more generally defined by the previous formulas.

Another observation is that the state of the brachistochrone (referring to the number N of photons) is directly related to the harmonic n (referring to the excited state of the photon),
$$n = \frac{\lambda_0}{\lambda_n},$$
$$N_0 = \frac{L_0}{\lambda_0}, \qquad N_n = \frac{L_0}{\lambda_n},$$
$$n = \frac{\lambda_0}{\lambda_n} = \frac{L_0}{\lambda_n} \frac{1}{N_0} = \frac{\lambda_0}{L_0} N_n$$

Also, the energy of the brachistochrone E_n at any harmonic n, is a multiple squared of the energy E_0 at the first harmonic,
$$E_n = n^2 E_0$$

This gives us the freedom to use the terms 'state,' or 'harmonic,' of the brachistochrone interchangeably, keeping in mind the distinction between the numbers n and N, wherever this is necessary.

Notes:
We have already used the assumption that if the speed v_0 of the object at the first state, $n=1$, is set equal to the speed of light c, then from the formulas,

$$v_n^2 = \frac{M_n}{m_0}c^2 = n^2\frac{M_0}{m_0}c^2, \qquad v_0^2 = \frac{M_0}{m_0}c^2, \qquad n = 1$$

$$v_0^2 = \frac{1}{2\pi}g_0 L_0$$

$$c^2 = \frac{1}{2\pi}\frac{m_0}{M_0}g_0 L_0$$

we take the simpler result,

$$v_0 = c \Leftrightarrow M_0 = m_0 \Rightarrow$$

$$v_n = n\sqrt{\frac{M_0}{m_0}}c = nc$$

Another possible way to simplify the energy equation is to assume that as the harmonic *n* increases, the number of photons N_n which compose the length L_0 approaches the number *n*,

$$n \gg 1, \qquad N_n \to n, \qquad N_0 \approx 1 \Rightarrow$$

$$L_0 \approx \lambda_0$$

By equating the length L_0 of the brachistochrone to the wavelength λ_0 of the reference photon, the number of photons *N* will always be equal to their harmonic *n*, so that we take simpler relationships, such as

$$E_0 = \frac{L_0}{\lambda_0}\varepsilon_0 = \frac{1}{2\pi}m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{\lambda_0^2}L_0 = \frac{L_0}{\lambda_0}\mu_0 c^2 = \frac{L_0}{\lambda_0^2}l_P m_P c^2$$

$$E_n = n^2 E_0 = \frac{1}{2\pi}m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2}L_0 = \frac{\lambda_0}{\lambda_n^2}L_0\mu_0 c^2 = \frac{L_0}{\lambda_n}\mu_n c^2 = \frac{l_P}{\lambda_n^2}L_0 m_P c^2$$

$$L_0 \approx \lambda_0 \Rightarrow$$

$$E_0 = \frac{L_0}{\lambda_0}\varepsilon_0 \equiv \varepsilon_0 = \frac{1}{2\pi}m_0 g_0 \lambda_0 = m_0 v_0^2 = \frac{hc}{\lambda_0} = \mu_0 c^2 = \frac{l_P}{\lambda_0}m_P c^2$$

$$E_n = n^2 E_0 = \frac{1}{2\pi}m_0 g_n \lambda_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2}\lambda_0 = \frac{\lambda_0}{\lambda_n}\mu_n c^2 = n\mu_n c^2 = \frac{l_P}{\lambda_n^2}\lambda_0 m_P c^2$$

$$L_0 = \frac{1}{2\pi}g_0 T_0^2 = v_0 T_0 \equiv c\tau_0$$

$$g_0 = 2\pi\frac{v_0^2}{L_0} \equiv 2\pi\frac{v_0^2}{\lambda_0}$$

$$T_0 = \sqrt{2\pi\frac{L_0}{g_0}} \equiv \sqrt{2\pi\frac{\lambda_0}{g_0}} = \sqrt{\frac{\lambda_0^2}{v_0^2}} = \sqrt{\frac{c^2\tau_0^2}{v_0^2}} = \frac{c}{v_0}\tau_0$$

$$T_n = \frac{1}{n}T_0 = \frac{1}{n}\frac{c}{v_0}\tau_0 = \frac{c}{v_0}\tau_n = \frac{c}{v_n}\tau_0, \qquad v_n = nv_0$$

$$v_0 = \frac{L_0}{T_0} \equiv c\frac{\tau_0}{T_0} = \sqrt{\frac{M_0}{m_0}}c \equiv \sqrt{\frac{\mu_0}{m_0}}c$$

$$v_n = \frac{L_0}{T_n} = nv_0 \equiv nc\frac{\tau_0}{T_0} = c\frac{\tau_0}{T_n} \equiv n\sqrt{\frac{\mu_0}{m_0}}c$$

If we additionally equate the speed v_0 to the speed of light c,

$$v_0 = c, \qquad M_0 \equiv \mu_0 = m_0,$$

$$L_0 = \lambda_0,$$

then we take even simpler expressions,

$$L_0 = \frac{1}{2\pi}g_0T_0^2 = v_0T_0 \equiv c\tau_0$$

$$g_0 = 2\pi\frac{v_0^2}{L_0} \equiv 2\pi\frac{v_0^2}{\lambda_0} \equiv 2\pi\frac{c^2}{\lambda_0}$$

$$g_n = 2\pi\frac{v_n^2}{L_0} \equiv 2\pi\frac{v_n^2}{\lambda_0} \equiv 2\pi n^2\frac{c^2}{\lambda_0}$$

$$T_0 = \frac{c}{v_0}\tau_0 \equiv \tau_0$$

$$T_n = \frac{1}{n}T_0 \equiv \frac{1}{n}\tau_0 = \tau_n$$

$$v_0 = c, \qquad v_n = nc$$

However, by doing this, from the energy equation we also take that,

$$E_0 = \frac{1}{2\pi}m_0g_0L_0 = 2\pi m_0\frac{GM_0}{L_0} = M_0c^2 = m_0v_0^2 = \frac{hc}{\lambda_0^2}L_0, \qquad g_0 = 4\pi^2\frac{GM_0}{L_0^2}$$

$$v_0 = c, \qquad M_0 = m_0, \qquad L_0 = \lambda_0 \Rightarrow$$

$$E_0 = 2\pi\frac{Gm_0^2}{\lambda_0} = m_0c^2 = \frac{hc}{\lambda_0} \Rightarrow$$

$$m_0 = \frac{1}{2\pi}\frac{c^2\lambda_0}{G} = \frac{h}{c\lambda_0},$$

$$\lambda_0 = \frac{h}{m_0 c} = 2\pi\frac{hG}{c^3\lambda_0} \Rightarrow$$

$$\lambda_0^2 = 2\pi\frac{hG}{c^3} = 4\pi^2\frac{\hbar G}{c^3} \Rightarrow$$

$$\lambda_0 = \sqrt{2\pi\frac{hG}{c^3}} = 2\pi\sqrt{\frac{\hbar G}{c^3}} = 2\pi r_P \equiv l_P,$$

$$l_P = 2\pi r_P = 2\pi\sqrt{\frac{\hbar G}{c^3}} = 2\pi\sqrt{\frac{1}{2\pi}\frac{hG}{c^3}} = \sqrt{2\pi\frac{hG}{c^3}},$$

$$m_0 = \frac{h}{c\lambda_0} \equiv \frac{h}{cl_P} \equiv m_P$$

so that we take as a result that the wavelength λ_0 of the reference photon will be equal to Planck length l_P, while the mass m_0 of the moving object will be equal to Planck mass m_P.

Thus, although by equating different parameters we may take simple results, oversimplifications may result in our losing the physical meaning of the problem.

Finally we may also note some aspects with respect to the term in the energy equation associated with Planck energy. By dividing the brachistochrone into Planck lengths l_P (instead of wavelengths λ_0 of some photon of reference), we take for the energy per wavelength, that

$$\varepsilon_0 = \frac{hc}{\lambda_0} = \mu_0 c^2 = \frac{\lambda_P}{\lambda_0} m_P c^2$$

$$\lambda_0 \equiv l_P \Rightarrow$$

$$\varepsilon_0 = \frac{hc}{\lambda_P} \equiv \varepsilon_P$$

so that the energy ε_0 per wavelength λ_0 will be identical to Planck energy ε_P.

In that sense the energy E_0 may be considered the total energy (for all Planck lengths l_P),

$$E_0 = \frac{1}{2\pi} m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{\lambda_0^2} L_0 = \frac{L_0}{\lambda_0} \mu_0 c^2 = \frac{L_0}{\lambda_0^2} l_P m_P c^2$$

$$\lambda_0 \equiv l_P \Rightarrow$$

$$E_0 = \frac{1}{2\pi} m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{l_P^2} L_0 = \frac{L_0}{l_P} \mu_0 c^2 = \frac{L_0}{l_P} m_P c^2$$

where, among others, we have the following relationships,

$$g_0 = 4\pi^2 \frac{GM_0}{L_0^2} = 2\pi \frac{v_0^2}{L_0} = 2\pi \frac{hc}{m_0 l_P^2} = 2\pi \frac{c^2}{l_P} \frac{m_P}{m_0}$$

$$v_0^2 = \frac{L_0}{l_P} \frac{m_P}{m_0} c^2$$

$$c^2 = \frac{hc}{l_P m_P}$$

An application of such a substitution will be further explored later on.

Still, if the total energy E_n, for all harmonics n, is n^2 times greater than the energy E_0 at the first state, $n=1$,

$$E_0 = \frac{hc}{l_P^2} L_0$$

$$E_n = n^2 E_0 \Rightarrow$$

$$E_n = \frac{hc}{\lambda_n^2} L_0 = n^2 \frac{hc}{\lambda_0^2} L_0 \equiv n^2 \frac{hc}{l_P^2} L_0, \qquad \lambda_0 \equiv l_P \Rightarrow$$

$$\lambda_n = \frac{1}{n} l_P$$

then we may assume photon wavelengths λ_n much shorter than Planck length l_P.

Therefore, in addition to the meaning of photons with macroscopic wavelengths, as we earlier saw, $\lambda_0 \approx L_0$, $\lambda_n << L_0$, we have also to consider the consequences of having photons with wavelengths smaller than Planck length, $\lambda_0 \approx l_P$, $\lambda_n << \lambda_P$.

However, in such cases, the problem may not lie on the theoretically infinite possibilities, but on the practical limits of measurement.

The photon and the graviton

We have already discussed, in the section about quantum entanglement, a possible model for the interaction between a brachiston of mass m, which travels on the brachistochrone S, and the reference photon of wavelength λ, which propagates on the linear axis L.

Here the result can be specialized, so that the object moving on the brachistochrone is a graviton, while the photon is related to the graviton either directly (the photon is emitted by the graviton), or indirectly (the photon is produced as a disturbance in spacetime, due to the graviton's motion).

The relationship between the graviton and the reference photon can be described from the energy equation,

$$E_0 = \frac{1}{2\pi} m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{\lambda_0^2} L_0 = \frac{L_0}{\lambda_0^2} l_P m_P c^2 = \frac{L_0}{\lambda_0^2} \lambda_g m_g c^2, \qquad n = 1$$

$$E_n = \frac{1}{2\pi} m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{l_P}{\lambda_n^2} L_0 m_P c^2 = \frac{\lambda_g}{\lambda_n^2} L_0 m_g c^2$$

The same equation can also be written in the variable form,

$$E_B = \frac{1}{2\pi} m g L_B = M_B c^2 = m v^2 = \frac{hc}{\lambda^2} L_B = \frac{l_P}{\lambda^2} L_B m_P c^2 = \frac{\lambda_g}{\lambda^2} L_B m_g c^2$$

where the index 'B' stands for '*Brachistochrone*', while the quantities E_B, g, M_B, v, λ are variables (the rest of the quantities being constant) with respect to the harmonic, or state, n.

If we replace in this energy equation the mass m of any object by the mass m_g of a graviton, then the same equation will describe the interaction between a graviton of mass m_g, and a photon of wavelength λ. Thus we have,

$m \equiv m_g \Rightarrow$

$$E_B = \frac{1}{2\pi} m_g g L_B = 2\pi \frac{G m_g M_B}{L_B} = M_B c^2 = m_g v^2 = \frac{hc}{\lambda^2} L_B = \frac{l_P}{\lambda^2} L_B m_P c^2 = \frac{\lambda_g}{\lambda^2} L_B m_g c^2$$

Comparing the different pairs in this energy equation, we can take a relationship between the wavelength λ of the reference photon, and the wavelength λ_g, or the mass m_g, of the graviton,

$$\lambda^2 = 2\pi \frac{c^2}{g} \lambda_g = \frac{m_g}{M_B} \lambda_g L_B = \frac{c^2}{v^2} \lambda_g L_B = \frac{1}{2\pi} \frac{\lambda_g c^2}{G M_B} L_B^2$$

Supposing, for example, that the graviton travels at the speed of light, we have

$$\lambda^2 = \frac{m_g}{M_B} \lambda_g L_B = \frac{c^2}{v^2} \lambda_g L_B$$

$$v = c, \qquad m_g = M_B \Rightarrow$$

$$\lambda^2 = \lambda_g L_B$$

More generally, the relationship between the mass m_g of the graviton, and the mass M_B of the brachistochrone, will be

$$M_B c^2 = \frac{\lambda_g}{\lambda^2} L_B m_g c^2 \Rightarrow$$

$$\frac{M_B}{m_g} = \frac{\lambda_g}{\lambda^2} L_B$$

As far as the final speed of the graviton is concerned, we have that

$$M_B c^2 = m_g v^2 \Rightarrow$$

$$v^2 = \frac{M_B}{m_g} c^2 = N' c^2,$$

$$N' = \frac{M_B}{m_g}$$

Therefore, we see that the speed v of the graviton will be $\sqrt{N'}$ times greater than the speed of light c, where N' is the number of gravitons of mass m_g which compose the brachistochrone of mass M_B.

As we shall see later on, even greater speeds can be attained by a smaller mass (that of a brachiston lighter than the graviton).

Notes:

If we equate in the previous relationships the mass M_B and the length L_B of the brachistochrone to the mass m_g and the wavelength λ_g of the graviton, respectively, we take that

$$E_B = \frac{1}{2\pi} m_g g L_B = 2\pi m_g \frac{GM_B}{L_B} = M_B c^2 = m_g v^2 = \frac{hc}{\lambda^2} L_B = \frac{l_P}{\lambda^2} L_B m_P c^2 = \frac{\lambda_g}{\lambda^2} L_B m_g c^2,$$

$$g = \frac{GM_B}{R_B^2} = 4\pi^2 \frac{GM_B}{L_B^2},$$

$$m_g = M_B, \qquad \lambda_g = L_B \quad \Rightarrow$$

$$E_B = \frac{1}{2\pi} m_g g \lambda_g = 2\pi \frac{Gm_g^2}{\lambda_g} = m_g c^2 = m_g v^2 = \frac{hc}{\lambda^2} \lambda_g = \frac{l_P}{\lambda^2} \lambda_g m_P c^2 = \frac{\lambda_g^2}{\lambda^2} m_g c^2,$$

$$m_g c^2 = m_g v^2 \Rightarrow$$

$$v = c,$$

$$m_g c^2 = \frac{\lambda_g^2}{\lambda^2} m_g c^2 \Rightarrow$$

$$\lambda = \lambda_g,$$

so that the energy equation is reduced into the following one,

$$E_B = 2\pi \frac{Gm_g^2}{\lambda_g} = m_g c^2 = \frac{hc}{\lambda_g} = \frac{l_P}{\lambda_g} m_P c^2$$

However, comparing the first and the last of the energy terms, we have that

$$2\pi \frac{Gm_g^2}{\lambda_g} = \frac{l_P}{\lambda^2} \lambda_g m_P c^2,$$

$$\lambda = \lambda_g \Rightarrow$$

$$2\pi \frac{Gm_g^2}{\lambda_g} = \frac{l_P}{\lambda_g} m_P c^2 \Rightarrow$$

$$2\pi G m_g^2 = l_P m_P c^2 = \frac{h}{c m_P} m_P c^2 = hc \Rightarrow$$

$$m_g^2 = \frac{1}{2\pi} \frac{hc}{G} = \frac{\hbar c}{G} \equiv m_P^2$$

$$\lambda_g^2 = \left(\frac{h}{cm_g}\right)^2 \equiv \left(\frac{h}{cm_P}\right)^2 \equiv l_P^2$$

Therefore such substitutions turn the brachistochrone into a micro- black hole (a Planck particle), whose length L_B will be equal to Planck length l_P, and whose mass M_B will be equal to Planck mass m_P.

In what follows we shall see that particles even smaller than a Planck particle can be obtained.

The graviton as the brachiston

As we have already said, the graviton may be considered as a special case of a brachiston. The hypothesis of the brachiston can be based on the aspect that masses smaller than the graviton mass are possible.

As far as the graviton is concerned, according to Wikipedia, its mass is estimated as follows:
The analysis of gravitational waves yielded a new upper bound on the mass of gravitons, if gravitons are massive at all.

The graviton's Compton wavelength

$$\lambda = \frac{h}{mc}$$

is at least 1.6×10^{16}m, or about 1.6 light-years, corresponding to a graviton mass of no more than 7.7×10^{-23}eV/c^2.

Recent observations of gravitational waves have put an upper bound of 1.2×10^{-22} eV/c^2 on the graviton's mass.
[https://en.wikipedia.org/wiki/Graviton]
[https://en.wikipedia.org/wiki/Compton_wavelength]

Transforming the units of the graviton mass into *kg*, we have

$$m_g = 1.2 \times 10^{-22}\frac{eV}{c^2} = \left(1.2 \times 10^{-22}\frac{eV}{c^2}\right)\left(1.782662 \times 10^{-36}\frac{kg}{eV/c^2}\right) = 2.139 \times 10^{-58}kg$$

This is the upper bound (thus the maximum) of the graviton mass, according to the recent observations.

If we put this value for the graviton mass m_g into Compton's formula for the wavelength, we take

$$\lambda_g = \frac{h}{m_g c} = \frac{2.335 \times 10^{-58}\frac{kgly^2}{y}}{(2.139 \times 10^{-58}kg)\left(1\frac{ly}{y}\right)} = 1.092ly$$

where we changed the units of h in light units, as follows

$$h = 6.626 \times 10^{-34} \frac{m^2 kg}{s} = 6.626 \times 10^{-34} \frac{m^2 kg}{s} \frac{3.154 \times 10^7 \frac{s}{y}}{\left(9.461 \times 10^{15} \frac{m}{ly}\right)^2} \Rightarrow$$

$$h = 2.335 \times 10^{-58} \frac{kgly^2}{y}$$

This will be the lower bound for the graviton's wavelength λ_g (minimum wavelength), corresponding to the upper bound of the graviton's mass m_g (maximum mass).

An interesting remark here is that the numerical value of Planck constant h (in light units) is almost identical to the value of the graviton mass m_g (in kg):

$$[h] = 2.335 \times 10^{-58},$$

$$[m_g] = 2.139 \times 10^{-58}$$

In fact, if the average wavelength λ_g of a graviton is exactly $1ly$, then these two values will numerically coincide:

$$\lambda_g = \frac{h}{m_g c},$$

$$\lambda_g = 1ly \Rightarrow$$

$$m_g = \frac{h}{\lambda_g c} = \frac{2.335 \times 10^{-58} \frac{kgly^2}{y}}{(1ly)\left(1 \frac{ly}{y}\right)} = 2.335 \times 10^{-58} kg,$$

$$[m_g] = [h] = 2.335 \times 10^{-58}, \qquad \lambda_g = 1ly$$

While there doesn't seem to be any specific reason why the wavelength λ_g of the graviton should be exactly equal to $1ly$, such an assumption may offer us a more accurate estimation for the mass m_g of the graviton (based on the numerical coincidence between Planck constant and the graviton mass).

Now taking the energy of the brachistochrone, and supposing that its mass M_B is equal to the mass of the graviton m_g, its radius R_B will be

$$m_g g R_B = m_g \frac{GM_B}{R_B} = M_B c^2$$

$$M_B = m_g \Rightarrow$$

$$\frac{Gm_g^2}{R_B} = m_g c^2 \Rightarrow$$

$$R_B = \frac{Gm_g}{c^2},$$

so that

$$R_B = \frac{\left(6.674 \times 10^{-11} \frac{m^3}{kgs^2}\right)(2.335 \times 10^{-58} kg)}{\left(3 \times 10^8 \frac{m}{s}\right)^2} = 5.195 \times 10^{-85} m$$

where we have used the value for the graviton's mass m_g corresponding to a graviton's wavelength $\lambda_g = 1ly$.

This result gives us the Schwarzschild's radius of a black hole with mass equal to the graviton mass.

The aspect here is that we have obtained a size much smaller than Planck radius.

However, a size even smaller can be taken. Supposing that the maximum possible value for a particle's wavelength can be equal to the radius of the observable universe, which we may call R_U, then the minimum possible mass, which we may call m_b, for a particle will be

$$m_b = \frac{h}{R_U c} = \frac{2.335 \times 10^{-58} \frac{kgly^2}{y}}{(1.380 \times 10^{10} ly)\left(1 \frac{ly}{y}\right)} = 1.692 \times 10^{-68} kg$$

where the radius R_U of the observable universe is assumed to be

$$R_U = 13.8Gly = 1.380 \times 10^{10} ly$$

The corresponding mass m_b will be the smallest possible mass in the observable universe.

On the other hand, if the maximum possible mass of a particle is the mass of the observable universe, which we may call M_U, then its corresponding wavelength, which we may call l_b, will be

$$l_b = \frac{h}{M_U c} = \frac{2.335 \times 10^{-58} \frac{kgly^2}{y}}{(1.760 \times 10^{53} kg)\left(1 \frac{ly}{y}\right)} = 1.327 \times 10^{-111} ly \Rightarrow$$

$$l_b = (1.327 \times 10^{-111} ly)\left(9.461 \times 10^{15} \frac{m}{ly}\right) = 1.255 \times 10^{-95} m$$

This will be the smallest possible size (wavelength) in the observable universe.

Incidentally, the value for the mass M_U of the observable universe which was used here is somewhat bigger than the current estimated value, and will be derived later on.

The notation l_b, instead of λ_b, was used in accordance to the notation for Planck length l_P (instead of λ_P), because the values m_b and l_b do not necessarily refer to the same particle, but to a range of values for different particles. Thus, using the notation (λ_g, m_g) for the wavelength and the mass respectively of the graviton, the particle (l_b, M_U) will be the smallest but heaviest particle, while the particle (R_U, m_b) will be the biggest but lightest particle in the universe. In this sense, all such 'particles,' from the lightest of them, to the universe itself, may be considered brachistons (thus the index '*b*').

If we now use the value m_b as the mass M_B of the brachistochrone, then the corresponding value for the radius R_B of the brachistochrone will be,

$$R_B = \frac{Gm_b}{c^2} = \frac{\left(6.674 \times 10^{-11} \frac{m^3}{kgs^2}\right)(1.692 \times 10^{-68} kg)}{\left(3 \times 10^8 \frac{m}{s}\right)^2} = 1.255 \times 10^{-95} m \equiv l_b$$

This result gives us the Schwarzschild's radius of a black hole with mass equal to a brachiston mass m_b. This mass, together with the corresponding size l_b, will presumably refer to the smallest and lightest possible object in the universe.

Here is a comparison between the two formulas which have been used in this section, referring to Schwarzschild radius R_S, and Compton wavelength, which we may also call λ_C, respectively:

$$R_S = \frac{GM_S}{c^2}$$

$$\lambda_C = \frac{h}{cm_C}$$

More general relationships can be taken for the same formulas by using the energy equation of the brachistochrone,

$$E_B = mgR_B = m\frac{GM_B}{R_B} = M_B c^2 = mv^2 = 2\pi\frac{hc}{\lambda^2}R_B, \qquad g = \frac{GM_B}{R_B^2} \Rightarrow$$

$$R_B = \frac{GM_B}{v^2} = \frac{Gm}{c^2} = \frac{1}{2\pi}\frac{M_B c}{h}\lambda^2 = \frac{1}{2\pi}\frac{mv^2}{hc}\lambda^2,$$

$$\lambda^2 = 2\pi\frac{h}{M_B c}R_B = 2\pi\frac{hc}{mv^2}R_B = 2\pi\frac{hc}{mv^2}\frac{Gm}{c^2} = 2\pi\frac{hG}{c}\frac{1}{v^2}$$

If in the first formula

$$R_B = \frac{GM_B}{v^2} = \frac{Gm}{c^2}$$

we equate the mass of the brachistochrone M_B to the mass of the moving object m, and we also equate the speed of the moving object v to the speed of light c, we take back Schwarzschild's formula.

If in the second formula

$$\lambda^2 = 2\pi\frac{h}{M_B c}R_B = 2\pi\frac{hc}{mv^2}R_B$$

we additionally equate the wavelength λ of the reference photon to the radius R_B (or to the length L_B) of the brachistochrone, we take back Compton's formula.

But, in the general case, while the radius R_B of the brachistochrone does not necessarily refer to a black hole, in the second formula the wavelength λ always refers to the size of the reference photon, not to the dimensions of the moving object of mass m.

Thus the notion of the brachistochrone can be used not only to bring together, but also to distinguish between the various aspects of the problem, either such aspects refer to black holes in the macrocosm, or to Compton wavelengths of particles in the microcosm.

Notes:

As we earlier divided the brachistochrone using the numbers N, N_P, and N', referring either to the number of photons, Planck lengths, or gravitons which the length L_B or the mass M_B of the brachistochrone is composed of respectively, the same can be done with respect to the number of brachistons, which we may call N'', which the mass M_B of the brachistochrone consists of. Thus we may expand the energy equation of the brachistochrone, as follows

$$E_n = \frac{1}{2\pi} m g_n L_0 = M_n c^2 = m v_n^2 = n^2 \frac{hc}{\lambda_0^2} L_0 = n^2 N_0 \mu_0 c^2 = n^2 N_P \frac{l_P^2}{\lambda_0^2} m_P c^2 = n^2 N_0{}' m_g c^2$$

$$= n^2 N_0{}'' m_b c^2 \Rightarrow$$

$$E_n = \frac{1}{2\pi} m g_n L_0 = M_n c^2 = m v_n^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{\lambda_0^2}{\lambda_n^2} N_0 \mu_0 c^2 = \frac{\lambda_0^2}{\lambda_n^2} N_P \frac{l_P^2}{\lambda_0^2} m_P c^2 = \frac{\lambda_0^2}{\lambda_n^2} N_0{}' m_g c^2$$

$$= \frac{\lambda_0^2}{\lambda_n^2} N_0{}'' m_b c^2 \Rightarrow$$

$$E_n = \frac{1}{2\pi} m g_n L_0 = M_n c^2 = m v_n^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{\lambda_0^2}{\lambda_n^2} \frac{L_0}{\lambda_0} \mu_0 c^2 = \frac{\lambda_0^2}{\lambda_n^2} \frac{L_0}{l_P} \frac{l_P^2}{\lambda_0^2} m_P c^2 = \frac{\lambda_0^2}{\lambda_n^2} \frac{\lambda_g}{\lambda_0^2} L_0 m_g c^2$$

$$= \frac{\lambda_0^2}{\lambda_n^2} \frac{L_0}{l_b} m_b c^2 = \frac{\lambda_0^2}{\lambda_n^2} \frac{L_0^2}{\lambda_0^2} m_b c^2 \Rightarrow$$

$$E_n = \frac{1}{2\pi} m g_n L_0 = M_n c^2 = m v_n^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{\lambda_0}{\lambda_n^2} L_0 \mu_0 c^2 = \frac{l_P}{\lambda_n^2} L_0 m_P c^2 = \frac{\lambda_g}{\lambda_n^2} L_0 m_g c^2$$

$$= \frac{L_0^2}{\lambda_n^2} m_b c^2 = \frac{\lambda_0^2}{\lambda_n^2} \frac{m_b}{l_b} L_0 c^2 = n^2 M_0 c^2$$

The same equation can be written using the variable notation (without the index *n*), as follows

$$E_B = \frac{1}{2\pi} mgL_B = M_B c^2 = mv^2 = \frac{hc}{\lambda^2} L_B = \frac{\lambda_0}{\lambda^2} L_B \mu_0 c^2 = \frac{l_P}{\lambda^2} L_B m_P c^2 = \frac{\lambda_g}{\lambda^2} L_B m_g c^2$$

$$= \frac{L_B^2}{\lambda^2} m_b c^2 = \frac{\lambda_0^2}{\lambda^2} \frac{m_b}{l_b} L_B c^2 = n^2 M_0 c^2$$

where

$$n = \frac{\lambda_0}{\lambda_n} \equiv \frac{\lambda_0}{\lambda},$$

$$N = \frac{L_B}{\lambda}, \qquad \lambda = \frac{1}{n}\lambda_0 = \frac{1}{n}\frac{h}{c\mu_0} = \frac{h}{c\mu}$$

$$N_P = \frac{L_B}{l_P}, \qquad l_P = \frac{h}{cm_P}$$

$$N' = \frac{M_B}{m_g} = \frac{h}{c\lambda^2}\frac{L_B}{m_g} = \frac{\lambda_g}{\lambda^2} L_B\,, \qquad M_B = \frac{h}{c\lambda^2} L_B, \qquad m_g = \frac{h}{c\lambda_g}$$

$$N'' = \frac{M_B}{m_b} = \frac{M_B c}{h} L_B = \frac{h}{c\lambda^2} L_B \frac{c}{h} L_B = \frac{h}{cl_b}\frac{c}{h} L_B = \frac{L_B^2}{\lambda^2} = \frac{L_B}{l_b},$$

$$m_b = \frac{h}{cL_B} = \frac{h}{c\lambda^2} l_b, \qquad l_b = \frac{h}{cM_B} = \frac{\lambda^2}{L_B}$$

The purpose here is to estimate the maximum value of the numbers N_P, N', and N'' at the highest possible state *n*, supposing that this state is given by substituting the radius R_B (or the length L_B) of the brachistochrone with the radius R_U of the observable universe, and the mass M_B of the brachistochrone with the mass M_U of the observable universe.

These are the related quantities,

$$M_U = 1.760 \times 10^{53} kg$$

$$R_U = 1.380 \times 10^{10} ly = (1.380 \times 10^{10} ly)\left(9.461 \times 10^{15} \frac{m}{ly}\right) = 1.306 \times 10^{26} m,$$

$$1 ly = 9.461 \times 10^{15} m$$

$$r_P = \frac{1}{2\pi} l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616 \times 10^{-35} m$$

$$m_P = \frac{h}{cl_P} = \sqrt{\frac{\hbar c}{G}} = 2.176 \times 10^{-8} kg$$

$$m_g = \frac{h}{c\lambda_g} = 2.335 \times 10^{-58} kg$$

$$\lambda_g = \frac{h}{cm_g} = 1ly = 9.461 \times 10^{15} m$$

$$m_b = \frac{h}{cR_U} = 1.692 \times 10^{-68} kg, \qquad R_U \equiv L_U$$

$$l_b = \frac{h}{cM_U} = 1.255 \times 10^{-95} m$$

Some attention is needed with respect to the factor of 2π, whether we refer to the radius R_U and the length L_U of the observable universe, or to Planck radius r_P and Planck length l_P. Also the quantities m_b and l_b in general depend on the radius R_B and the mass M_B of the brachistochrone, respectively.

Thus we have,

$$n = \frac{\lambda_0}{\lambda_n} \equiv \frac{\lambda_0}{\lambda},$$

$$N = \frac{R_U}{\lambda}, \qquad \lambda = \frac{1}{n}\lambda_0 = \frac{1}{n}\frac{h}{c\mu_0} = \frac{h}{c\mu}$$

$$N_P = \frac{R_U}{l_P} = \frac{M_U}{m_P} = \frac{1.306 \times 10^{26} m}{1.616 \times 10^{-35} m} = \frac{1.760 \times 10^{53} kg}{2.176 \times 10^{-8} kg} = 0.8082 \times 10^{61}$$

$$N' = \frac{M_U}{m_g} = \frac{1.760 \times 10^{53} kg}{2.335 \times 10^{-58} kg} = 0.7538 \times 10^{111}$$

$$N'' = \frac{M_U}{m_b} = \frac{R_U}{l_b} = \frac{1.760 \times 10^{53} kg}{1.692 \times 10^{-68} kg} = \frac{1.306 \times 10^{26} m}{1.255 \times 10^{-95} m} = 1.041 \times 10^{121}$$

Therefore the radius R_U of the observable universe consists of $N_P \approx 10^{61}$ Planck lengths, while the mass M_U of the observable universe is composed of $N' \approx 10^{111}$ graviton masses m_g, or $N'' \approx 10^{121}$ brachiston masses m_b.

Additionally, we have the following ratios,

$$\mathrm{N} = \frac{N}{N_P} = \frac{R_U}{\lambda}\frac{l_P}{R_U} = \frac{l_P}{\lambda}$$

$$\mathrm{N}' = \frac{N'}{N_P} = \frac{M_U}{m_g}\frac{l_P}{R_U} = \frac{M_U}{m_g}\frac{m_P}{M_U} = \frac{m_P}{m_g} = \frac{0.7538 \times 10^{111}}{0.8082 \times 10^{61}} = 0.9327 \times 10^{50}$$

$$\mathrm{N}'' = \frac{N''}{N_P} = \frac{M_U}{m_b}\frac{l_P}{R_U} = \frac{M_U}{m_b}\frac{m_P}{M_U} = \frac{m_P}{m_b} = \frac{l_P}{l_b} = \frac{1.041 \times 10^{121}}{0.8082 \times 10^{61}} = 1.288 \times 10^{60}$$

Since the related quantities are constants, the ratios $\mathcal{N}'$ and $\mathcal{N}''$ will also be constant.

Here we can mention the following interesting things. Firstly, taking the ratio,

$$\rho = \frac{M_U}{R_U} = \frac{m_P}{l_P} = \frac{m_b}{l_b} = \frac{1.760 \times 10^{53} kg}{1.306 \times 10^{26} m} = \frac{2.176 \times 10^{-8} kg}{1.616 \times 10^{-35} m} = \frac{1.692 \times 10^{-68} kg}{1.255 \times 10^{-95} m} \Rightarrow$$

$$\rho = 1.348 \times 10^{27}\frac{kg}{m}$$

we see that the value for the mass M_U of the observable universe can be taken from first principles (supposing, for example, that the radius R_U of the observable universe, as well as the values for Planck mass m_P and Planck length l_P, are given). Incidentally, the quantity ρ has units of linear density.

Secondly, taking the ratio

$$\frac{\mathrm{N}''}{\mathrm{N}'} = \frac{m_g}{m_b} = \frac{R_U}{\lambda_g} = \frac{1.288 \times 10^{60}}{0.9327 \times 10^{50}} = 1.381 \times 10^{10} \equiv [R_U],$$

$$\lambda_g \equiv 1ly \Rightarrow$$

$$m_g = \frac{m_b R_U}{\lambda_g} = \frac{hR_U}{cR_U\lambda_g} = \frac{h}{c\lambda_g} \Rightarrow$$

$$[m_g] = [h], \qquad [c] = [\lambda_g] = 1$$

we see that if this ratio numerically coincides with the radius R_U of the observable universe, then the mass m_g and the wavelength λ_g of the graviton will numerically coincide with Planck constant h and the speed of light c, respectively.

Finally, with respect to the speed, if we suppose that the object moving on the brachistochrone is a brachiston of mass m_b, while the brachistochrone is the universe itself, then from the energy equation we have that,

$$E_B = M_B c^2 = m v^2$$

$$M_B \equiv M_U, \qquad m \equiv m_b \Rightarrow$$

$$M_U c^2 = m_b v^2 \Rightarrow$$

$$v^2 = \frac{M_U}{m_b} c^2 = N'' c^2, \qquad N'' = 1.041 \times 10^{121} \Rightarrow$$

$$v \approx 10^{60} c$$

Presumably, this will be the maximum possible speed in the observable universe.

The universe as a brachistochrone

Apart from the numerical coincidences which we saw in the previous section, if in the energy equation of the brachistochrone we replace the mass M_B of the brachistochrone with the mass M_U of the observable universe, and the radius R_B of the brachistochrone with the radius R_U of the observable universe, then we take

$$E_B = mgR_B = m\frac{GM_B}{R_B} = M_B c^2 = mv^2 = \frac{hc}{\lambda^2}L_B = \frac{\lambda_0}{\lambda^2}L_B\mu_0 c^2 = \frac{l_P}{\lambda^2}L_B m_P c^2 = \frac{\lambda_g}{\lambda^2}L_B m_g c^2$$

$$M_B \equiv M_U, \qquad L_B \equiv R_U \Rightarrow$$

$$M_U c^2 = \frac{hc}{\lambda^2}R_U \Rightarrow$$

$$\lambda^2 = \frac{h}{c}\frac{R_U}{M_U} = \frac{h}{c}\frac{l_P}{m_P} = \frac{1}{\rho}\frac{h}{c} \equiv l_P^2,$$

$$m_P = \frac{h}{cl_P}, \qquad \rho = \frac{M_U}{R_U} \equiv \frac{m_P}{l_P}$$

where the ratio ρ, giving the linear density of the observable universe, was identified in the previous section.

Thus we take as result that the wavelength λ of the reference photon is reduced into Planck length l_P.

This result is not obvious, but it is a consequence of the notion of the brachistochrone, and of the energy equation which brings together a pair of phenomenally unrelated quantities, the ratio of the mass M_U of the observable universe to Planck mass m_P, or, equivalently, the ratio of the radius R_U of the observable universe to Planck length l_P,

$$E_B = M_U c^2 = \frac{hc}{l_P^2}R_U \Rightarrow$$

$$M_U = \frac{h}{cl_P^2}R_U = \frac{h}{cl_P}\frac{R_U}{l_P} = m_P\frac{R_U}{l_P},$$

$$\frac{M_U}{m_P} = \frac{R_U}{l_P} = N_P$$

The number N_P is a fixed quantity, which we have already calculated in the previous section.

But if we ignore the distinction between the different masses or lengths which appear in the latter formula, then the energy equation reduces to the following undifferentiated form,

$$E = mc^2 = \frac{h}{cl_P}$$

and loses its meaning.

Therefore the hypothesis of the brachistochrone is significant in order to take more general results.

In fact such results may offer us a better estimation for the mass M_U of the observable universe. The current estimate for this mass is

$$M_U = 1.5 \times 10^{53} kg$$

[https://en.wikipedia.org/wiki/Observable_universe]

But here we have that

$$\frac{M_U}{m_P} = \frac{R_U}{l_P} \Rightarrow$$

$$M_U = \frac{R_U}{l_P} m_P = \frac{1.306 \times 10^{26} m}{1.616 \times 10^{-35} m} 2.176 \times 10^{-8} kg = 1.759 \times 10^{53} kg$$

If we accept this result as more accurate, then the brachistochrone hypothesis can be used in order to give physical meaning to the numerical coincidence of the previous ratio.

Another consequence of such a coincidence is the following one. We have already talked about Schwarzschild radius, which is given by the formula

$$R_S = \frac{2GM_S}{c^2}$$

where R_S is the radius of a black hole, and M_S is its mass.

This formula can be derived from the mechanical energy of a system, if this energy is gravitational and kinetic energy. Supposing, for example, that an object of mass *m* leaves the surface of the Earth with a speed *v*, and is set in orbit around the Earth, then if *R* is the radius of the orbit, and M_E is the mass of the Earth, the conservation of the mechanical energy gives

$$mgR = \frac{1}{2}mv^2 \Rightarrow$$

$$m\frac{GM_E}{R^2}R = \frac{1}{2}mv^2, \qquad g = \frac{GM_E}{R^2} \Rightarrow$$

$$m\frac{GM_E}{R} = \frac{1}{2}mv^2 \Rightarrow$$

$$v^2 = \frac{2GM_E}{R},$$

$$R = \frac{2GM_E}{v^2}$$

The speed *v* is the escape speed of the object from Earth's gravity. If this speed was equal to the speed of light *c*, then the Earth would turn into a black hole, and its radius would be given by Schwarzschild's formula, replacing *v* by *c* in the previous equation,

$$R = \frac{2GM_E}{v^2}$$

$$v = c \Rightarrow$$

$$R_S = \frac{2GM_E}{c^2} = \frac{2\left(6.674 \times 10^{-11}\frac{m^3}{kgs^2}\right)(5.972 \times 10^{24}kg)}{\left(3 \times 10^8\frac{m}{s}\right)^2} = 8.857 \times 10^{-3}m$$

This would be about 10 millimeters.

Now we will see what happens if we replace in Schwarzschild's formula the mass of the gravitating body with the mass M_U of the observable universe.

It would also be helpful to change the units of Newton's gravitational constant *G* into light units, as follows

$$G = 6.674 \times 10^{-11} \frac{m^3}{kgs^2} = 6.674 \times 10^{-11} \frac{m^3}{kgs^2} \left(\frac{1}{9.461 \times 10^{15} \frac{m}{ly}} \right)^3 \left(3.154 \times 10^7 \frac{s}{y} \right)^2 \Rightarrow$$

$$G = 7.840 \times 10^{-44} \frac{ly^3}{kgy^2}$$

Replacing these values into Schwarzschild's formula (omitting the factor of 2), we take

$$R_S = \frac{GM_U}{c^2} = \frac{\left(7.840 \times 10^{-44} \frac{ly^3}{kgy^2}\right)(1.760 \times 10^{53} kg)}{\left(1 \frac{ly}{y}\right)^2} = 1.380 \times 10^{10} ly \equiv R_U$$

Remarkably enough, the value of Schwarzschild radius R_S in this case is identical to the radius R_U of the observable universe. But since the Schwarzschild radius R_S of the universe is (exactly or about) equal to the radius R_U of the observable universe, then we must conclude that the universe is a black hole!

An explanation of this coincidence could be that the linear density ρ of the observable universe, which we may also call ρ_U, is constant,

$$\rho_U = \frac{M_U}{R_U} = \frac{m_P}{l_P} = \frac{1.760 \times 10^{53} kg}{1.306 \times 10^{26} m} = \frac{1.760 \times 10^{53} kg}{1.380 \times 10^{10} ly} = \frac{2.176 \times 10^{-8} kg}{1.616 \times 10^{-35} m} \Rightarrow$$

$$\rho_U = 1.348 \times 10^{27} \frac{kg}{m} = 1.275 \times 10^{43} \frac{kg}{ly},$$

$$M_U = 1.760 \times 10^{53} kg,$$

$$R_U = 1.380 \times 10^{10} ly = 1.306 \times 10^{26} m$$

An observation here is that the (inverse) linear density of the universe ρ_U numerically coincides with the gravitational constant G (in units of light),

$$G = 6.674 \times 10^{-11} \frac{m^3}{kgs^2} = 7.838 \times 10^{-44} \frac{ly^3}{kgy^2}$$

$$\rho_U = 1.348 \times 10^{27} \frac{kg}{m} = 1.275 \times 10^{43} \frac{kg}{ly},$$

$$\frac{1}{\rho_U} = 7.843 \times 10^{-44}\,\frac{ly}{kg},$$

$$[G] \equiv \left[\frac{1}{\rho_U}\right] = \left[\frac{R_U}{M_U}\right]$$

Calculating the product of these two quantities, we have

$$\rho_U G = \left(1.275 \times 10^{43}\,\frac{kg}{ly}\right)\left(7.840 \times 10^{-44}\,\frac{ly^3}{kg y^2}\right) = 0.9996\,\frac{ly^2}{y^2} = 1\,\frac{ly^2}{y^2} \equiv c^2$$

$$\rho_U G = c^2 = const.$$

Therefore, returning to Schwarzschild radius, we take that

$$R_S = \frac{GM_U}{c^2} = \frac{G\rho_U R_U}{c^2} \equiv \frac{c^2 R_U}{c^2} \equiv R_U$$

Now another interesting aspect arises if we take the following two ratios, which we also saw in the previous section,

$$[h] \equiv [m_g] = \left[\frac{h}{c\lambda_g}\right] \Rightarrow$$

$$[\lambda_g] \equiv [c]$$

$$\frac{Ɲ''}{Ɲ'} = \frac{R_U}{\lambda_g} \equiv [R_U], \qquad [\lambda_g] \equiv 1$$

If the wavelength λ_g of the graviton numerically coincides with the speed of light c, then we can write that

$$\frac{R_U}{\lambda_g} = \frac{M_U}{\rho_U \lambda_g} = \frac{GM_U}{c^2 \lambda_g}, \qquad \rho_U = \frac{M_U}{R_U} = \frac{c^2}{G} \Rightarrow$$

$$\frac{\rho_U G}{\lambda_g} = \frac{c^2}{\lambda_g} = \frac{1(ly/y)^2}{1ly} = 1\,\frac{ly}{y^2} \equiv \mathscr{g},$$

$$c^2 = \mathscr{g}\lambda_g$$

Presumably, the quantity $\mathscr{g}$ will be constant throughout the universe, since all the related quantities are constants, and will be equal to $1ly/y^2$.

But the coincidences do not end here. Another observation we can make is that that acceleration of gravity, on the Earth's surface, which we may call g_E, is approximately equal to $1ly/y^2$, if we change the units into light units as follows,

$$g_E = 9.820\frac{m}{s^2} = \left(9.820\frac{m}{s^2}\right)\left(\frac{1}{9.461\times 10^{15}}\frac{ly}{m}\right)\left(3.154\times 10^7\frac{s}{y}\right)^2 = 1.033\frac{ly}{y^2} \approx g$$

$$1ly = 9.461\times 10^{15}m$$

$$1y = 3.154\times 10^7 s$$

$$g = 1\frac{ly}{y^2}$$

Another way to express this coincidence is with respect to the linear density ρ or area density σ of the Earth and of the universe. Given the values for the mass M_E and radius R_E of the Earth, and for the mass M_U and radius R_U of the observable universe, respectively,

$$M_E = 5.972\times 10^{24}kg$$

$$R_E = 6.371\times 10^6 m$$

$$M_U = 1.760\times 10^{53}kg$$

$$R_U = 1.306\times 10^{26}m$$

for the linear density ρ_E and area density σ_E of the Earth we have,

$$\rho_E = \frac{M_E}{R_E} = \frac{5.972\times 10^{24}kg}{6.371\times 10^6 m} = 0.9374\times 10^{18}\frac{kg}{m}$$

$$\sigma_E = \frac{M_E}{R_E^2} = \frac{\rho_E}{R_E} = \frac{0.9374\times\frac{10^{18}kg}{m}}{6.371\times 10^6 m} = 1.472\times 10^{11}\frac{kg}{m^2}$$

while for the linear density ρ_U and area density σ_U of the observable universe we have

$$\rho_U = \frac{M_U}{R_U} = \frac{1.760\times 10^{53}kg}{1.306\times 10^{26}m} = 1.348\times 10^{27}\frac{kg}{m}$$

$$\sigma_U = \frac{M_U}{R_U^2} = \frac{\rho_U}{R_U} = \frac{1.760\times 10^{53}kg/m}{(1.306\times 10^{26}m)^2} = 10.32\frac{kg}{m^2}$$

We can also calculate the corresponding accelerations for those densities,

$$\mathcal{g} = \frac{\rho_U G}{\lambda_g} = \frac{GM_U}{R_U}\frac{1}{\lambda_g} = \frac{\sigma_U G R_U}{\lambda_g} = \frac{g_U R_U}{\lambda_g}\frac{c^2}{\lambda_g} = \frac{c^2}{\lambda_g} = 1\frac{ly}{y^2}$$

$$g_U = \frac{GM_U}{R_U^2} = \frac{G\rho_U}{R_U} = \frac{c^2}{R_U} = \sigma_U G = \frac{(1\ ly/y)^2}{1.380 \times 10^{10} ly} = 0.7246 \times 10^{-10}\frac{ly}{y^2}$$

$$g_E = \frac{GM_E}{R_E^2} = \frac{G\rho_E}{R_E} = \sigma_E G$$

so that if we identify the acceleration on the Earth's surface g_E with the acceleration $\mathcal{g}$, it will be

$$\mathcal{g} = \frac{\rho_U G}{\lambda_g} = \frac{M_U G}{R_U}\frac{1}{\lambda_g} = \frac{\sigma_U G R_U}{\lambda_g} = \frac{g_U R_U}{\lambda_g} = \frac{GM_E}{R_E^2} = \frac{\rho_E G}{R_E} = \sigma_E G \equiv g_E$$

Incidentally, the ratio

$$\frac{\sigma_U G}{R_U} = \frac{\rho_U G}{R_U^2} = \frac{c^2}{R_U^2} = \frac{g_U}{R_U} = \omega_U^2$$

which has units of angular frequency, gives us Hubble constant,

$$\omega_U^2 = \frac{0.7246 \times 10^{-10}\frac{ly}{y^2}}{1.380 \times 10^{10} ly} = \frac{6.891 \times 10^{-10}\frac{m}{s^2}}{1.306 \times 10^{26} m} = 0.5251 \times 10^{-20}\frac{1}{y^2} = 5.277 \times 10^{-36}\frac{1}{s^2} \Rightarrow$$

$$\omega_U = 0.7247 \times 10^{-10}\frac{1}{y} = 2.297 \times 10^{-18}\frac{1}{s}$$

Hubble constant is estimated to be

$$H \approx 70\frac{km}{s \times Mpc},$$

[https://map.gsfc.nasa.gov/universe/uni_expansion.html]

where

$$1Mpc = 1 \times 10^6 pc = (1 \times 10^6 pc)\left(3.262\frac{ly}{pc}\right) = 3.262 \times 10^6 ly$$

$$= (3.262 \times 10^6 ly)\left(9.461 \times 10^{15}\frac{m}{ly}\right) = 3.087 \times 10^{22} m$$

so that,

$$H \approx 70 \frac{km}{s \times Mpc} = 70 \left(\frac{km}{s}\right) \frac{1 \times 10^3 \frac{m}{km}}{(3.087 \times 10^{22} m)} = 2.268 \times 10^{-18} \frac{1}{s}$$

Therefore the value of

$$\omega_U = 2.297 \times 10^{-18} \frac{1}{s}$$

can be seen as a more accurate estimation of Hubble constant.

Conclusively, from the equation for the acceleration $\mathscr{g}$,

$$\mathscr{g} = \frac{c^2}{\lambda_g} = \frac{\rho_U G}{\lambda_g} = \frac{M_U G}{R_U} \frac{1}{\lambda_g} = \frac{\sigma_U G R_U}{\lambda_g} = \frac{g_U R_U}{\lambda_g} = \frac{G M_E}{R_E^2} = \frac{\rho_E G}{R_E} = \sigma_E G \equiv g_E$$

we have the equivalent expressions,

$$g_E \lambda_g = g_U R_U = c^2$$

$$\frac{\rho_E}{R_E} = \frac{\rho_U}{\lambda_g}$$

$$\sigma_E = \frac{\sigma_U R_U}{\lambda_g} = \frac{\mathscr{g}}{G}$$

We should mention here that in the calculation of the area density σ we have assumed that the Earth was flat, or perhaps that the Earth's mass was spread across the surface of a sphere of radius equal to the Earth's radius (while we have ignored the radian factor π for simplicity). Such assumptions can be based on the holographic principle, which we will explore later on.

The same assumptions can also be based on the brachistochrone hypothesis. As the universe, so the Earth can be seen as a brachistochrone whose mass is either spread across the curve of the brachistochrone, or concentrated on the focus of the brachistochrone. This is as far as the linear density is concerned.

With respect to the area density, we have to suppose that the mass of the brachistochrone is spread within the area of the brachistochrone, thus the region enclosed by the rim L and the curve S of the brachistochrone. The area A of the brachistochrone is in fact, $A=3\pi R^2$, thus analogous to the radius of the brachistochrone squared.

Therefore, ultimately, approximating either the Earth or the universe with a brachistochrone, we take meaningful results which, besides their implication with respect to the appearance of life on Earth (considering for example that the acceleration of $1ly/y^2$ is ideal for the appearance of life as we know it), apply everywhere.

Incidentally, the acceleration g, which is constant throughout the universe, and (as has been treated here) equal to $1ly/y^2$, could be identified with a field, yet unknown.

Part 3

The driven harmonic oscillator

Up till now we have seen that the motion of an object in real space and time is an oscillatory motion, if spacetime is a medium which oscillates, so that the motion of the object can be described by the equations of the (damped) harmonic oscillator.

A more complete equation of motion can be taken if we introduce into the system a driving force. Commonly such a force is treated as sinusoidal, of the form

$F(t) = F_0 \cos \omega t$

Adding this term to the equation of the damped harmonic oscillator, we have

$ma + bv + ky = F_0 \cos \omega t$

whereas if the driving force is not present,

$F = 0 \Rightarrow$

$ma + bv + ky = 0$

we take back the (homogeneous) equation of the damped unforced harmonic oscillator.

The angular frequency ω of the driving force is different from the natural angular frequency ω_0 of the simple harmonic oscillator, and it is different from the angular frequency ω' of the damped harmonic oscillator. The angular frequency ω is imposed by an 'external' object onto the system of the oscillator, so that finally the whole system will oscillate with the frequency ω of the driving force.

This can be seen assuming a solution of the general form

$y(t) = C_1 e^{-\frac{\gamma}{2}t} \cos(\omega' t \pm \varphi') + C_2 \cos(\omega t \pm \varphi)$

where the amplitudes C_1 and C_2 are given by the following expressions,

$$C_1 = \frac{F_0}{m}\frac{\omega_0^2 - \omega^2}{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}, \qquad C_2 = \frac{F_0}{m}\frac{1}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}}$$

The first term of the solution refers to the damped (unforced) oscillator, while the second term refers to the driven oscillator. The former term is regarded as the transient term, because of the negative exponential which vanishes with time. The latter term is called the steady state solution, as the system thereafter oscillates with the frequency ω of the driving force.

Thus at the steady state we have

$$y(t) = y_0(\omega)\cos(\omega t - \varphi),$$

where

$$\tan\varphi = \frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

$$y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}}$$

The negative sign before the phase φ in the solution is a matter of choice. The constant C_2 here is replaced by the amplitude $y_0(\omega)$, which explicitly expresses its dependence on the angular frequency ω of the driving force.

The previous values for the amplitude $y_0(\omega)$, and for the phase φ, are taken by substituting the steady state solution into the equation of motion, as follows

$$ma + ky + bv = F_0\cos\omega t$$

$$y(t) = y_0(\omega)\cos(\omega t - \varphi)$$

$$ma + ky + bv = F_0\cos\omega t \Rightarrow$$

$$-m\omega^2 y_0(\omega)\cos(\omega t - \varphi) + ky_0(\omega)\cos(\omega t - \varphi) - b\omega y_0(\omega)\sin(\omega t - \varphi) =$$

$$F_0\cos\omega t$$

Using the identities

$\cos(\omega t - \varphi) = \cos\omega t \cos\varphi + \sin\omega t \sin\varphi$

$\sin(\omega t - \varphi) = \sin\omega t \cos\varphi - \cos\omega t \sin\varphi$

we take,

$ma + ky + bv = F_0 \cos\omega t \Rightarrow$

$-m\omega^2 y_0(\omega) \cos\omega t \cos\varphi - m\omega^2 y_0(\omega) \sin\omega t \sin\varphi$

$+ky_0(\omega) \cos\omega t \cos\varphi + ky_0(\omega) \sin\omega t \sin\varphi$

$-b\omega y_0(\omega) \sin\omega t \cos\varphi + b\omega y_0(\omega) \cos\omega t \sin\varphi =$

$F_0 \cos\omega t \Rightarrow$

$-m\omega^2 y_0(\omega) \sin\omega t \sin\varphi + ky_0(\omega) \sin\omega t \sin\varphi - b\omega y_0(\omega) \sin\omega t \cos\varphi =$

$F_0 \cos\omega t + m\omega^2 y_0(\omega) \cos\omega t \cos\varphi - ky_0(\omega) \cos\omega t \cos\varphi - b\omega y_0(\omega) \cos\omega t \sin\varphi = 0,$

so that if both sides of the previous equation are equal to zero, then the previous equation is true.

Taking the two sides separately and equating them to zero, we have,

$-m\omega^2 y_0(\omega) \sin\omega t \sin\varphi + ky_0(\omega) \sin\omega t \sin\varphi - b\omega y_0(\omega) \sin\omega t \cos\varphi = 0 \Rightarrow$

$-m\omega^2 y_0(\omega) \sin\varphi + ky_0(\omega) \sin\varphi - b\omega y_0(\omega) \cos\varphi = 0 \Rightarrow$

$-m\omega^2 \sin\varphi + k \sin\varphi - b\omega \cos\varphi = 0 \Rightarrow$

$-m\omega^2 \sin\varphi + m\omega_0^2 \sin\varphi - \gamma m\omega \cos\varphi = 0, \qquad k = m\omega_0^2, \qquad b = \gamma m \Rightarrow$

$-\omega^2 \sin\varphi + \omega_0^2 \sin\varphi - \gamma\omega \cos\varphi = 0 \Rightarrow$

$-\omega^2 \sin\varphi + \omega_0^2 \sin\varphi = \gamma m\omega \cos\varphi = 0 \Rightarrow$

$(\omega_0^2 - \omega^2) \sin\varphi = \gamma\omega \cos\varphi = 0 \Rightarrow$

$$\frac{\sin\varphi}{\cos\varphi} = \frac{\gamma\omega}{\omega_0^2 - \omega^2} \Rightarrow$$

$$\tan\varphi = \frac{\gamma\omega}{\omega_0^2 - \omega^2}, \qquad \varphi = \tan^{-1}\frac{\gamma\omega}{\omega_0^2 - \omega^2}$$

This expression gives the phase φ.

Now, equating the second term to zero, we take

$$F_0 \cos \omega t + m\omega^2 y_0(\omega) \cos \omega t \cos \varphi - k y_0(\omega) \cos \omega t \cos \varphi - b\omega y_0(\omega) \cos \omega t \sin \varphi = 0 \Rightarrow$$

$$F_0 + m\omega^2 y_0(\omega) \cos \varphi - k y_0(\omega) \cos \varphi - b\omega y_0(\omega) \sin \varphi = 0 \Rightarrow$$

$$F_0 + m\omega^2 y_0(\omega) \cos \varphi - m\omega_0^2 y_0(\omega) \cos \varphi - \gamma m\omega y_0(\omega) \sin \varphi = 0,$$

$$k = m\omega_0^2, \qquad b = \gamma m \Rightarrow$$

$$\frac{F_0}{m} + \omega^2 y_0(\omega) \cos \varphi - \omega_0^2 y_0(\omega) \cos \varphi - \gamma\omega y_0(\omega) \sin \varphi = 0 \Rightarrow$$

$$\frac{F_0}{m} = \omega_0^2 y_0(\omega) \cos \varphi - \omega^2 y_0(\omega) \cos \varphi + \gamma\omega y_0(\omega) \sin \varphi \Rightarrow$$

$$\frac{F_0}{m} = y_0(\omega)[(\omega_0^2 - \omega^2) \cos \varphi + \gamma\omega \sin \varphi]$$

The values of *sinφ* and *cosφ*, with respect to the expression for *tanφ* we previously found, are given as follows,

$$\sin^2 \varphi = 1 - \cos^2 \varphi = 1 - \frac{\sin^2 \varphi}{\tan^2 \varphi} \Rightarrow \sin^2 \varphi + \frac{\sin^2 \varphi}{\tan^2 \varphi} = 1 \Rightarrow \sin^2 \varphi \left(1 + \frac{1}{\tan^2 \varphi}\right) = 1 \Rightarrow$$

$$\sin \varphi = \sqrt{\frac{1}{1 + \frac{1}{\tan^2 \varphi}}} = \sqrt{\frac{1}{1 + \frac{(\omega_0^2 - \omega^2)^2}{\gamma^2 \omega^2}}} = \frac{\gamma\omega}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}}$$

$$\cos^2 \varphi = 1 - \sin^2 \varphi = 1 - \frac{\tan^2 \varphi}{1 + \tan^2 \varphi} = \frac{1}{1 + \tan^2 \varphi} \Rightarrow$$

$$\cos \varphi = \sqrt{\frac{1}{1 + \tan^2 \varphi}} = \sqrt{\frac{1}{1 + \frac{\gamma^2\omega^2}{(\omega_0^2 - \omega^2)^2}}} = \frac{\omega_0^2 - \omega^2}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}}$$

so that

$$\frac{F_0}{m} = y_0(\omega)[(\omega_0^2 - \omega^2) \cos \varphi + \gamma\omega \sin \varphi] \Rightarrow$$

$$\frac{F_0}{m} = y_0(\omega) \left[(\omega_0^2 - \omega^2) \frac{\omega_0^2 - \omega^2}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}} + \gamma\omega \frac{\gamma\omega}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}}\right],$$

$$\cos \varphi = \frac{\omega_0^2 - \omega^2}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}}, \qquad \sin \varphi = \frac{\gamma\omega}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \quad \Rightarrow$$

$$\frac{F_0}{m} = y_0(\omega)\left[\frac{(\omega_0^2-\omega^2)^2}{\sqrt{\gamma^2\omega^2+(\omega_0^2-\omega^2)^2}} + \frac{(\gamma\omega)^2}{\sqrt{\gamma^2\omega^2+(\omega_0^2-\omega^2)^2}}\right] \Rightarrow$$

$$\frac{F_0}{m} = y_0(\omega)\left[\frac{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}{\sqrt{\gamma^2\omega^2+(\omega_0^2-\omega^2)^2}}\right] = y_0(\omega)\sqrt{\gamma^2\omega^2+(\omega_0^2-\omega^2)^2} \Rightarrow$$

$$y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\gamma^2\omega^2+(\omega_0^2-\omega^2)^2}}$$

This expression gives the amplitude $y_0(\omega)$.

Thus, the steady state solution

$$y(t) = y_0(\omega)\cos(\omega t - \varphi)$$

is valid as long as

$$\tan\varphi = \frac{\gamma\omega}{\omega_0^2-\omega^2}, \qquad y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\gamma^2\omega^2+(\omega_0^2-\omega^2)^2}}$$

Taking into account the previous solution, for the speed and the acceleration of the oscillator we have,

$$ma + bv + ky = F_0\cos\omega t$$

$$y(t) = y_0(\omega)\cos(\omega t - \varphi)$$

$$v(t) = -\omega y_0(\omega)\sin(\omega t-\varphi) = -v_0(\omega)\sin(\omega t-\varphi), \qquad v_0(\omega) = \omega y_0(\omega)$$

$$a(t) = -\omega^2 y_0(\omega)\cos(\omega t-\varphi) = -a_0(\omega)\cos(\omega t-\varphi), \qquad a_0(\omega) = \omega^2 y_0(\omega) = \omega v_0(\omega)$$

The values for the amplitude $y_0(\omega)$ at $\omega=0$, and $\omega=\omega_0$, are respectively,

$$y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\gamma^2\omega^2+(\omega_0^2-\omega^2)^2}},$$

$$y_0(\omega=0) = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}, \qquad \omega_0^2 = \frac{k}{m}$$

$$y_0(\omega=\omega_0) = \frac{F_0}{m\gamma\omega_0} = \frac{F_0}{b\omega_0}, \qquad \gamma = \frac{b}{m}$$

The state at which the angular frequency ω of the driving force reaches the natural angular frequency ω_0 of the system, $\omega=\omega_0$, is called resonance.

Although at the steady state the transient term (which is due to damping) has disappeared, the behavior of the system still depends on the damping factor γ, which makes its appearance in the expression for the amplitude $y_0(\omega)$.

The maximum amplitude $y_0(\omega)_{max}$ can be taken if we differentiate the amplitude $y_0(\omega)$ with respect to the frequency ω, and set the differential equal to zero,

$$\frac{dy_0(\omega)}{d\omega} = 0 \Rightarrow$$

$$\frac{d}{d\omega}\frac{F_0/m}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}} = 0 \Rightarrow$$

$$\frac{d}{d\omega}\frac{1}{[\gamma^2\ \omega^2 + (\omega_0^2 - \omega^2)^2]^{1/2}} = 0, \qquad \frac{F_0}{m} \neq 0 \Rightarrow$$

$$\frac{d}{d\omega}[\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2]^{-1/2} = 0 \Rightarrow$$

$$-\frac{1}{2}[\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2]^{-\frac{3}{2}}\frac{d}{d\omega}[\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2] = 0 \Rightarrow$$

$$\frac{d}{d\omega}[\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2] = 0, \qquad -\frac{1}{2}[\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2]^{-\frac{3}{2}} \neq 0 \Rightarrow$$

$$2\gamma^2\omega + 2(\omega_0^2 - \omega^2)(-2\omega) = 0 \Rightarrow$$

$$2\gamma^2\omega - 4\omega(\omega_0^2 - \omega^2) = 0 \Rightarrow$$

$$2\gamma^2\omega = 4\omega(\omega_0^2 - \omega^2) \Rightarrow$$

$$\gamma^2 = 2(\omega_0^2 - \omega^2) \Rightarrow$$

$$\frac{\gamma^2}{2} = \omega_0^2 - \omega^2 \Rightarrow$$

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{2}$$

Inserting this value for the angular frequency ω back into the expression for the amplitude, we have

$$y_0(\omega) = \frac{F_0/m}{\sqrt{\gamma^2\,\omega^2 + (\omega_0^2 - \omega^2)^2}},$$

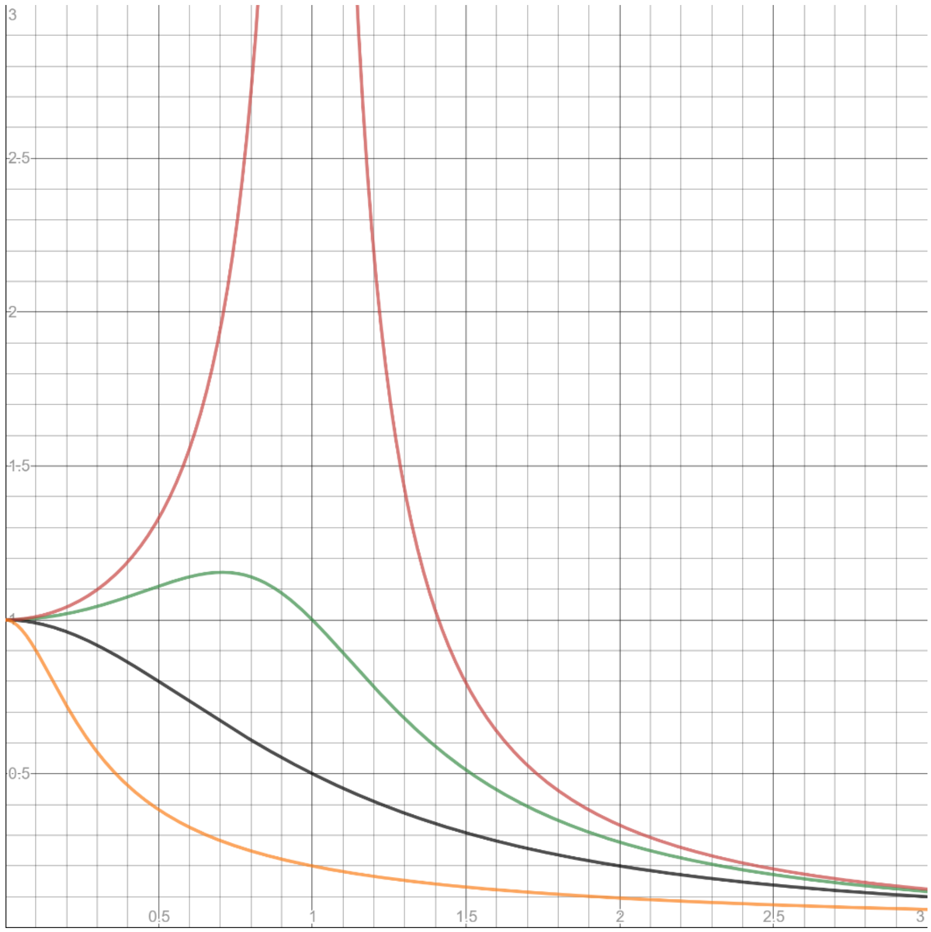
$$y_0(\omega)_{max} = \frac{F_0/m}{\sqrt{\gamma^2\,\omega^2 + \frac{\gamma^4}{4}}} = \frac{F_0/m}{\sqrt{\gamma^2\,(\omega_0^2 - \frac{\gamma^2}{2}) + \frac{\gamma^4}{4}}} = \frac{F_0/m}{\sqrt{\gamma^2\,\omega_0^2 - \frac{\gamma^4}{2} + \frac{\gamma^4}{4}}} \Rightarrow$$

$$y_0(\omega)_{max} = \frac{F_0/m}{\sqrt{\gamma^2\,\omega_0^2 - \frac{\gamma^4}{4}}} = \frac{F_0}{m\gamma\omega_0}\frac{1}{\sqrt{1 - \frac{\gamma^2}{4\omega_0^2}}} = y_0(\omega_0)\frac{1}{\sqrt{1 - \frac{\gamma^2}{4\omega_0^2}}}$$

From the last expression we see that the maximum amplitude can only be defined if $\gamma \neq 0$, or $\gamma \neq 2\omega_0$.

Therefore resonance is achieved at equal frequencies, $\omega \approx \omega_0$, only if the damping factor is sufficiently small $\gamma \approx 0$. However, if the damping factor γ is equal to zero, then the amplitude $y_0(\omega)$ will become infinite.

This is a graph of the amplitude $y_0(\omega)$, for different values of the damping factor γ,



$$y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \approx \frac{1}{\sqrt{\gamma^2\omega^2 + (1-\omega^2)^2}}, \qquad \omega_0 = 1, \qquad \frac{F_0}{m} = 1$$

Red: $y_0(\omega) \approx \frac{1}{\sqrt{0.01\omega^2+(1-\omega^2)^2}}, \ \gamma = 0.1$

Green: $y_0(\omega) \approx \frac{1}{\sqrt{\omega^2+(1-\omega^2)^2}}, \ \gamma = 1$

Black: $y_0(\omega) \approx \frac{1}{\sqrt{4\omega^2+(1-\omega^2)^2}}, \ \gamma = 2$

Orange: $y_0(\omega) \approx \frac{1}{\sqrt{25\omega^2+(1-\omega^2)^2}}, \ \gamma = 5$

As the graph shows, the smaller the damping factor γ is, the steeper the slope of the amplitude $y_0(\omega)$ will be. If the damping factor goes to zero, then the amplitude goes to infinity. If the angular frequency ω goes to infinity, then the amplitude goes to zero. Those marginal conditions, as well

as the condition $\gamma=2\omega_0$ (which is the condition of critical damping), will be examined in what follows.

As far as the energy of the driven and damped harmonic oscillator is concerned, the mechanical energy E_m can be taken from the steady state solution,

$$ma + ky + bv = F_0 \cos \omega t$$
$$y(t) = y_0(\omega) \cos \omega t$$
$$v(t) = -\omega y_0(\omega) \sin \omega t$$
$$a(t) = -\omega^2 y_0(\omega) \cos \omega t,$$

where for simplicity we set the phase φ equal to zero.

The elastic energy will be

$$E_{el}(t) = \frac{1}{2} k y^2(t) = \frac{1}{2} k y_0^2(\omega) \cos^2 \omega t = \frac{1}{2} m \omega_0^2 y_0^2(\omega) \cos^2 \omega t,$$

while the kinetic energy will be

$$E_k(t) = \frac{1}{2} m v^2(t) = \frac{1}{2} m \omega^2 y_0^2(\omega) \sin^2 \omega t$$

so that for the mechanical energy, we have that

$$E_m(t) = E_{el}(t) + E_k(t) = \frac{1}{2} m \omega_0^2 y_0^2(\omega) \cos^2 \omega t + \frac{1}{2} m \omega^2 y_0^2(\omega) \sin^2 \omega t$$

Choosing boundary conditions,

$$t = 0, \qquad \omega = 0,$$
$$\tan \varphi = \frac{\gamma \omega}{\omega_0^2 - \omega^2} = 0, \qquad \varphi = 0 \Rightarrow$$
$$y(t=0) = y_0(\omega = 0), \qquad v(t=0) = 0, \qquad a(t=0) = 0$$
$$F(t=0) = F_0 = k y_0(\omega = 0)$$

and

$$t = t_0 = T, \qquad \omega = \omega_0,$$

$$\tan\varphi = \infty, \qquad \varphi = \frac{\pi}{2} \Rightarrow$$

$$\cos(\omega t - \varphi) = \cos\left(\frac{2\pi}{T}t - \frac{\pi}{2}\right) = \cos\left(2\pi - \frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0,$$

$$\sin(\omega t - \varphi) = \sin\left(\frac{3\pi}{2}\right) = -1,$$

$$\cos\omega t = \cos\frac{2\pi}{T}t = \cos 2\pi = 1 \Rightarrow$$

$$y(T) = 0, \qquad v(T) = \omega_0 y_0(\omega_0) = v_0(\omega_0), \qquad a(T) = 0,$$

$$F_0 = F(t = T) = bv_0(\omega_0),$$

where

$$ky_0(0) = bv_0(\omega_0)$$

and using the values for the amplitude at $\omega=0$, or $\omega=\omega_0$,

$$y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}},$$

$$y_0(\omega = 0) = \frac{F_0}{m\omega_0^2}$$

$$y_0(\omega = \omega_0) = \frac{F_0}{m\gamma\omega_0}$$

we can find an expression for the mechanical energy at the boundaries,

$$E_m(t) = E_{el}(t) + E_k(t) = \frac{1}{2}m\omega_0^2 y_0^2(\omega)\cos^2(\omega t - \varphi) + \frac{1}{2}m\omega^2 y_0^2(\omega)\sin^2(\omega t - \varphi),$$

$$E_m(t = 0) = E_{el}(t = 0) + E_k(t = 0) \Rightarrow$$

$$E_m(t = 0) = \frac{1}{2}m\omega_0^2 y_0^2(\omega = 0) = \frac{1}{2}m\omega_0^2\left(\frac{F_0}{m\omega_0^2}\right)^2 = \frac{1}{2}m\omega_0^2\frac{F_0^2}{m^2\omega_0^4} \Rightarrow$$

$$E_m(t = 0) = \frac{F_0^2}{2m\omega_0^2}$$

$$E_m(t = T) = E_{el}(t = T) + E_k(t = T) \Rightarrow$$

$$E_m(t=T) = \frac{1}{2}m\omega_0^2 y_0^2(\omega=\omega_0) = \frac{1}{2}m\omega_0^2\left(\frac{F_0}{m\gamma\omega_0}\right)^2 = \frac{1}{2}m\omega_0^2\frac{F_0^2}{m^2\gamma^2\omega_0^2} \Rightarrow$$

$$E_m(t=T) = \frac{F_0^2}{2m\gamma^2}$$

where the quantity

$$E_0 \equiv E_m(t=0) = \frac{F_0^2}{2m\omega_0^2}$$

will represent the total energy of the oscillator.

In order to plot the graph of the mechanical energy $E_m(t)$ with respect to the driving angular frequency ω, thus the amplitude of the mechanical energy $E_m(\omega)$, it is convenient to eliminate the sinusoidal terms by taking the average value of the mechanical energy $<E_m(t)>$,

$$E_m(t) = E_{el}(t) + E_k(t) = \frac{1}{2}m\omega_0^2 y_0^2(\omega)\cos^2(\omega t-\varphi) + \frac{1}{2}m\omega^2 y_0^2(\omega)\sin^2(\omega t-\varphi),$$

$$\langle E_m(t)\rangle = \langle E_{el}(t)\rangle + \langle E_k(t)\rangle = \frac{1}{2}m\omega_0^2 y_0^2(\omega)\langle\cos^2(\omega t-\varphi)\rangle + \frac{1}{2}m\omega^2 y_0^2(\omega)\langle\sin^2(\omega t-\varphi)\rangle$$

$$\Rightarrow$$

$$\langle E_m(t)\rangle = \frac{1}{2}m\omega_0^2 y_0^2(\omega)\langle\cos^2\theta\rangle + \frac{1}{2}m\omega^2 y_0^2(\omega)\langle\sin^2\theta\rangle,$$

where we have set

$$\theta = \omega t - \varphi$$

The average value of a function is defined as follows,

$$\langle f(x)\rangle = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{x}\int_0^x f(x)dx\,, \qquad a=0, \qquad b=x$$

The average values of the functions $sin^2\theta$ and $cos^2\theta$, $\theta=\omega t+\varphi$, can be calculated using the following identities,

$$\cos 2\theta = \cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = 1 - \sin^2\theta \Rightarrow$$

$$\cos 2\theta = \cos^2\theta - (1-\cos^2\theta) = 2\cos^2\theta - 1 \Rightarrow 2\cos^2\theta = 1 + \cos 2\theta \Rightarrow$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta),$$

$$\cos 2\theta = (1-\sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta \Rightarrow 2\sin^2\theta = 1 - \cos 2\theta \Rightarrow$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

so that

$$\langle\sin^2\theta\rangle = \frac{1}{\theta}\int_0^\theta \sin^2\theta\, d\theta = \frac{1}{\theta}\int_0^\theta \frac{1}{2}(1-\cos 2\theta)d\theta = \frac{1}{\theta}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right] \Rightarrow$$

$$\langle\sin^2\theta\rangle = \frac{1}{2}, \qquad \theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

and

$$\langle\cos^2\theta\rangle = \langle 1 - \sin^2\theta\rangle = 1 - \frac{1}{2} = \frac{1}{2} = \langle\sin^2\theta\rangle$$

Thus the average value of the mechanical energy will be,

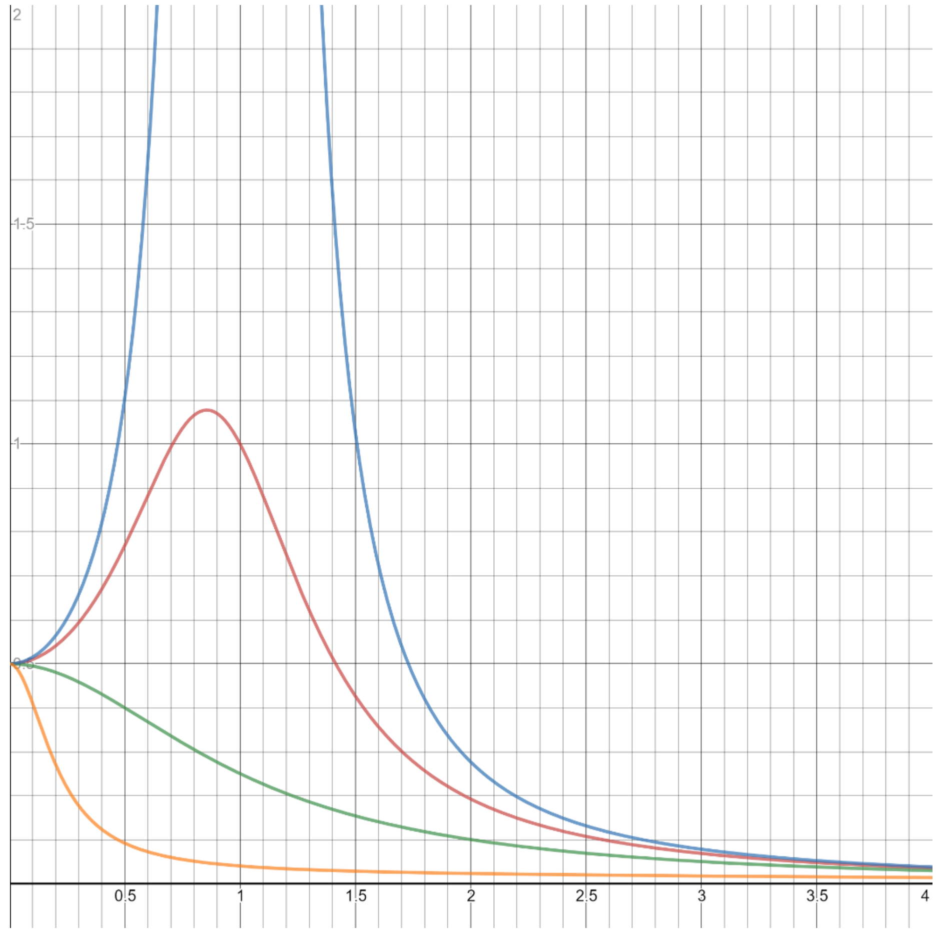
$$\langle E_m(t)\rangle = \frac{1}{2}m\omega_0^2 y_0^2(\omega)\langle\cos^2\theta\rangle + \frac{1}{2}m\omega^2 y_0^2(\omega)\langle\sin^2\theta\rangle, \qquad \theta = \omega t + \varphi \Rightarrow$$

$$\langle E_m(t)\rangle = \frac{1}{4}m\omega_0^2 y_0^2(\omega) + \frac{1}{4}m\omega^2 y_0^2(\omega) = \frac{1}{4}m y_0^2(\omega)(\omega_0^2 + \omega^2) \Rightarrow$$

$$\langle E_m(t)\rangle = \frac{1}{4}m\frac{F_0^2}{m^2}\frac{1}{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}(\omega_0^2 + \omega^2), \qquad y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \Rightarrow$$

$$\langle E_m(t)\rangle \equiv \langle E_m(\omega)\rangle = \frac{F_0^2}{4m}\frac{(\omega_0^2 + \omega^2)}{\gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}$$

This is a graph of the average mechanical energy $<E_m(t)>$, thus the average energy amplitude $<E_m(\omega)>$, of the driven and damped harmonic oscillator, for different values of the damping factor γ,



$$\langle E_m(\omega)\rangle = \frac{F_0^2}{4m}\frac{(\omega_0^2+\omega^2)}{\gamma^2\omega^2+(\omega_0^2-\omega^2)^2} \approx \frac{1}{2}\frac{(1+\omega^2)}{\gamma^2\omega^2+(1-\omega^2)^2}, \qquad \frac{F_0^2}{2m}=1, \qquad \omega_0 = 1$$

Blue: $\langle E_m(\omega)\rangle \approx \frac{1}{2}\frac{(1+\omega^2)}{0.01\omega^2+(1-\omega^2)^2},\ \gamma = 0.1$

Red: $\langle E_m(\omega)\rangle \approx \frac{1}{2}\frac{(1+\omega^2)}{\omega^2+(1-\omega^2)^2},\ \gamma = 1$

Green: $\langle E_m(\omega)\rangle \approx \frac{1}{2}\frac{(1+\omega^2)}{4\omega^2+(1-\omega^2)^2},\ \gamma = 2$

Orange: $\langle E_m(\omega)\rangle \approx \frac{1}{2}\frac{(1+\omega^2)}{25\omega^2+(1-\omega^2)^2},\ \gamma = 5$

This graph for the average energy amplitude $<E_m(\omega)>$ is similar to the one we saw for the amplitude $y_0(\omega)$, since the mechanical energy depends on the amplitude squared.

The factor of ½ was added because the initial (total) mechanical energy $E_m(t=0)$ is twice its average value $<E_m(\omega=0)>$,

$$E_m(t=0) = \frac{F_0^2}{2m\omega_0^2}, \qquad \omega = 0$$

$$\langle E_m(\omega = 0)\rangle = \frac{F_0^2}{4m\omega_0^2}$$

while the mechanical energy $E_m(t=T)$ at the other boundary, $t=T$, $\omega=\omega_0$, is the same as the average value $<E_m(\omega=\omega_0)>$,

$$E_m(t=T) = \frac{F_0^2}{2m\gamma^2}, \qquad \omega = \omega_0$$

$$\langle E_m(\omega_0)\rangle = \frac{F_0^2}{2m\gamma^2}$$

In fact, the latter energy, at $t=T$ (according to the boundary conditions we chose earlier), is purely kinetic energy, so that if the damping factor γ is comparable to the natural angular frequency ω_0, $\gamma\approx\omega_0\approx 1$, then the total energy (the initial elastic energy E_{el}) of the oscillator will have transformed into the kinetic energy E_k at resonance, $\omega\approx\omega_0$, $\omega\approx\omega_0\approx\gamma\approx 1$, so that

$$E_0 = E_m(t=0) = \frac{F_0^2}{2m\omega_0^2} \equiv E_{el}, \qquad \omega = 0$$

$$E_m(t=T) = \frac{F_0^2}{2m\gamma^2} \equiv E_k, \qquad \omega = \omega_0$$

$$\gamma \approx \omega_0 \Rightarrow$$

$$E_k \equiv E_0$$

A more general relationship between the driving angular frequency ω and the damping factor γ will be established later on.

Notes:

A common approximation for the mechanical energy $E_m(t)$ is taken if we suppose that the damping factor γ is sufficiently small, $\gamma\approx 0$, so that it will also be $\omega\approx\omega_0$. This can be seen from the formula which gives the maximum amplitude,

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{2},$$

$\gamma \approx 0 \Rightarrow \omega \approx \omega_0$

However this does not necessarily mean that the driving force has achieved resonance, because, as we shall see, the damping factor γ may depend on the angular frequency ω of the driving force.

In any case, if the angular frequency ω of the driving force is close to the natural angular frequency ω_0 of the oscillator, in order to further manipulate the expression for the average mechanical energy,

$$\langle E_m(t)\rangle = \frac{F_0^2}{4m}\frac{(\omega_0^2+\omega^2)}{\gamma^2\omega^2+(\omega_0^2-\omega^2)^2}$$

we may use the following approximations,

$$\omega_0+\omega \approx \omega_0+\omega_0 = 2\omega_0$$
$$\omega_0^2+\omega^2 \approx \omega_0^2+\omega_0^2 = 2\omega_0^2$$
$$\omega_0^2-\omega^2 = (\omega_0-\omega)(\omega_0+\omega) \approx (\omega_0-\omega)2\omega_0,$$
$$(\omega_0^2-\omega^2)^2 \approx 4\omega_0^2(\omega_0-\omega)^2,$$
$$\gamma \approx 0, \qquad \omega \approx \omega_0$$

so that the expression for the average mechanical energy takes the form

$$\langle E_m(t)\rangle = \frac{F_0^2}{4m}\frac{(\omega_0^2+\omega^2)}{\gamma^2\omega^2+(\omega_0^2-\omega^2)^2} \approx \frac{F_0^2}{4m}\frac{2\omega_0^2}{\gamma^2\omega_0^2+4\omega_0^2(\omega_0-\omega)^2} \Rightarrow$$
$$\langle E_m(t)\rangle = \frac{F_0^2}{2m}\frac{1}{\gamma^2+4(\omega_0-\omega)^2}, \qquad \gamma \approx 0, \qquad \omega \approx \omega_0$$

which has a maximum at $\omega_0=\omega$,

$$\langle E_m(\omega)\rangle_{max} = \frac{F_0^2}{m\gamma^2} = \frac{mF_0^2}{b^2}, \qquad \gamma = \frac{b}{m}$$

However, if $\gamma \approx 0$, then the maximum amplitude of the average mechanical energy of the system will be infinite,

$$\langle E_m(\omega)\rangle_{max} = \frac{F_0^2}{m\gamma^2} \Rightarrow$$

$$\langle E_m(\omega)\rangle_{max} = \infty, \quad \gamma = 0$$

Thus the mechanical energy of the system cannot be defined if the damping factor is zero, $\gamma \approx 0$.

The variable damping factor

The fact that the mechanical energy of the driven and damped harmonic oscillator cannot be defined (is infinite) if the damping factor γ is zero, gives us an indication that the damping factor may depend on the angular frequency ω of the driving force.

An expression with respect to a variable damping factor $\gamma(\omega)$ can be taken directly from the formula for the amplitude $y_0(\omega)$, as follows

$$y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\gamma_0^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \Rightarrow$$

$$y_0(\omega) = \frac{F_0}{m\gamma_0\omega}\frac{1}{\sqrt{1 + \frac{1}{\gamma_0^2}\frac{(\omega_0^2 - \omega^2)^2}{\omega^2}}} = \frac{F_0}{m\omega}\frac{1}{\gamma_0\sqrt{1 + \frac{1}{\gamma_0^2}\frac{(\omega_0^2 - \omega^2)^2}{\omega^2}}} = \frac{F_0}{m\gamma(\omega)\omega} \Rightarrow$$

$$\gamma(\omega) = \gamma_0\sqrt{1 + \frac{1}{\gamma_0^2}\frac{(\omega_0^2 - \omega^2)^2}{\omega^2}}, \qquad y_0(\omega) = \frac{F_0}{m\gamma(\omega)\omega}$$

where we have supposed that the damping factor γ_0 within the expression for the amplitude $y_0(\omega)$ corresponds to $\omega=\omega_0$,

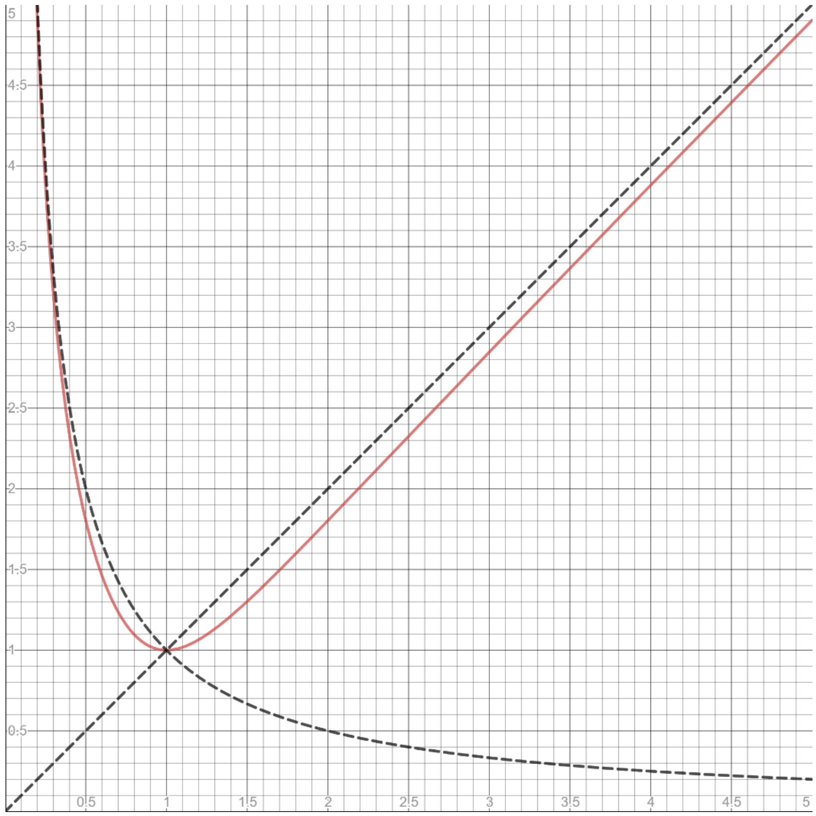
$$\gamma(\omega) = \gamma_0\sqrt{1 + \frac{1}{\gamma_0^2}\frac{(\omega_0^2 - \omega^2)^2}{\omega^2}},$$

$$\omega = \omega_0 \Rightarrow$$

$$\gamma(\omega_0) \equiv \gamma_0$$

In that sense, at resonance, $\omega=\omega_0$, the damping factor $\gamma(\omega_0)$ will not be zero, but equal to the value γ_0.

This is graph of the damping factor $\gamma(\omega)$, with respect to the angular frequency ω of the driving force:



$$\gamma(\omega)=\gamma_0\sqrt{1+\frac{1}{\gamma_0^2}\frac{(\omega_0^2-\omega^2)^2}{\omega^2}}\approx\sqrt{1+\frac{(1-\omega^2)^2}{\omega^2}},\qquad \gamma_0\approx\omega_0=1$$

The choice of $\gamma_0=\omega_0$ is the simplest one (for example we may have set $\gamma_0=2\omega_0$). The black dotted lines (of the form $y=x$ and $y=1/x$) were plotted for comparison.

The important aspect of the previous graph is that the damping factor $\gamma(\omega)$ takes its minimum value at γ_0, where $\omega=\omega_0$, instead of becoming zero. If $\omega<\omega_0$, then $\gamma(\omega)$ behaves like $1/\omega$, while if $\omega>\omega_0$, then $\gamma(\omega)$ behaves like ω (dotted lines in the graph).

The relationship between the damping factor $\gamma(\omega)$ and the amplitude $y_0(\omega)$ is the following one,

$$y_0(\omega)=\frac{F_0}{m}\frac{1}{\sqrt{\gamma_0^2\omega^2+(\omega_0^2-\omega^2)^2}},\qquad y_0(0)=\frac{F_0}{m\omega_0^2},\qquad y_0(\omega_0)=\frac{F_0}{m\gamma_0\omega_0}\Rightarrow$$

$$y_0(\omega)=\frac{F_0}{m\gamma(\omega)\omega}=\frac{F_0}{m}\frac{\omega_0^2}{\omega_0^2}\frac{1}{\gamma(\omega)\omega}=\frac{F_0}{m\omega_0^2}\frac{\omega_0^2}{\gamma(\omega)\omega}=y_0(0)\frac{\omega_0^2}{\gamma(\omega)\omega}$$

and

$$y_0(\omega) = \frac{F_0}{m\gamma(\omega)\omega} = \frac{F_0}{m}\frac{\gamma_0\omega_0}{\gamma_0\omega_0}\frac{1}{\gamma(\omega)\omega} = \frac{F_0}{m\gamma_0\omega_0}\frac{\gamma_0\omega_0}{\gamma(\omega)\omega} = y_0(\omega_0)\frac{\gamma_0\omega_0}{\gamma(\omega)\omega},$$

where

$$y_0(0)\frac{\omega_0^2}{\gamma(\omega)\omega} = y_0(\omega_0)\frac{\gamma_0\omega_0}{\gamma(\omega)\omega} \Rightarrow$$

$$\frac{y_0(0)}{y_0(\omega_0)} = \frac{\gamma_0\omega_0}{\gamma(\omega)\omega}\frac{\gamma(\omega)\omega}{\omega_0^2} = \frac{\gamma_0}{\omega_0}$$

Here we will derive an expression for the energy of the system with respect to the damping factor $\gamma(\omega)$, if the driving angular frequency ω of the moving object is greater than the natural angular frequency ω_0 of the oscillator, $\omega >> \omega_0$.

In such a case, from the equation which gives the mechanical energy,

$$E_m(t) = E_{el}(t) + E_k(t) = \frac{1}{2}m\omega_0^2 y_0^2(\omega)\cos^2(\omega t - \varphi) + \frac{1}{2}m\omega^2 y_0^2(\omega)\sin^2(\omega t - \varphi)$$

we may write,

$$E_m(t) = E_{el}(\omega)\cos^2(\omega t - \varphi) + E_k(\omega)\sin^2(\omega t - \varphi),$$

where

$$E_{el}(\omega) = \frac{1}{2}m\omega_0^2 y_0^2(\omega) = \frac{1}{2}m\omega_0^2\frac{F_0^2}{m^2\gamma^2(\omega)\omega^2} = \frac{F_0^2\omega_0^2}{2m\gamma^2(\omega)\omega^2},$$

$$E_k(\omega) = \frac{1}{2}m\omega^2 y_0^2(\omega) = \frac{1}{2}m\omega^2\frac{F_0^2}{m^2\gamma^2(\omega)\omega^2} = \frac{F_0^2}{2m\gamma^2(\omega)},$$

$$y_0(\omega) = \frac{F_0}{m\gamma(\omega)\omega} \Rightarrow$$

so that we may set

$$E_m(\omega) = E_{el}(\omega) + E_k(\omega) = \frac{1}{2}m\omega_0^2 y_0^2(\omega) + \frac{1}{2}m\omega^2 y_0^2(\omega) \Rightarrow$$

$$E_m(\omega) = \frac{F_0^2 \omega_0^2}{2m\gamma^2(\omega)\omega^2} + \frac{F_0^2}{2m\gamma^2(\omega)} = \frac{F_0^2}{2m\gamma^2(\omega)}\left(1 + \frac{\omega_0^2}{\omega^2}\right)$$

We may notice here that the energy amplitude $E_m(\omega)$ is twice as much as the average energy amplitude $< E_m(\omega) >$

$$\langle E_m(t)\rangle \equiv \langle E_m(\omega)\rangle = \frac{1}{4}m\omega_0^2 y_0^2(\omega) + \frac{1}{4}m\omega^2 y_0^2(\omega) = \frac{1}{4}my_0^2(\omega)(\omega_0^2 + \omega^2) \Rightarrow$$

$$\langle E_m(\omega)\rangle = \frac{1}{4}m\frac{F_0^2}{m^2\gamma^2(\omega)\omega^2}(\omega_0^2 + \omega^2) = \frac{F_0^2}{4m\gamma^2(\omega)}\frac{\omega_0^2 + \omega^2}{\omega^2} = \frac{F_0^2}{4m\gamma^2(\omega)}\left(1 + \frac{\omega_0^2}{\omega^2}\right) \Rightarrow$$

$$\langle E_m(\omega)\rangle = \frac{1}{2}\left[\frac{F_0^2}{4m\gamma^2(\omega)}\left(1 + \frac{\omega_0^2}{\omega^2}\right)\right] = \frac{1}{2}E_m(\omega)$$

Now if the angular frequency ω of the driving force is sufficiently larger than the natural frequency ω_0 of the oscillator, then we have that

$$\omega \gg \omega_0, \qquad \gamma(\omega) \approx \omega \Rightarrow$$

$$E_m(\omega) = \frac{F_0^2}{2m\gamma^2(\omega)}\left(1 + \frac{\omega_0^2}{\omega^2}\right) \approx \frac{F_0^2}{2m\gamma^2(\omega)} \Rightarrow$$

$$E_m(\omega) \equiv E_k(\omega) \approx \frac{F_0^2}{2m\gamma^2(\omega)} \approx \frac{F_0^2}{2m\omega^2}$$

and

$$E_m(t) \equiv E_k(t) = E_k(\omega)\sin^2(\omega t - \varphi) \approx \frac{F_0^2}{2m\gamma^2(\omega)}\sin^2(\omega t - \varphi) \approx \frac{F_0^2}{2m\omega^2}\sin^2(\omega t - \varphi)$$

The same expression can be directly taken from the expression for the mechanical energy $E_m(t)$, if $\omega >> \omega_0$,

$$E_m(t) = E_{el}(t) + E_k(t) = \frac{1}{2}m\omega_0^2 y_0^2(\omega)\cos^2(\omega t - \varphi) + \frac{1}{2}m\omega^2 y_0^2(\omega)\sin^2(\omega t - \varphi),$$

$$\omega \gg \omega_0 \Rightarrow$$

$$E_m(t) \approx \frac{1}{2}m\omega^2 y_0^2(\omega)\sin^2(\omega t - \varphi) = E_k(t) \Rightarrow$$

$$E_m(t) \approx E_k(t) = E_k(\omega)\sin^2(\omega t - \varphi) = \frac{F_0^2}{2m\gamma^2(\omega)}\sin^2(\omega t - \varphi),$$

$$E_k(\omega) = \frac{F_0^2}{2m\gamma^2(\omega)} \approx \frac{F_0^2}{2m\omega^2}, \qquad \omega \gg \omega_0, \qquad \gamma(\omega) \approx \omega$$

Thus, if the angular frequency ω of the driving force is sufficiently larger than the natural frequency ω_0 of the oscillator, then the mechanical energy of the system is purely kinetic energy.

In other words, if we associate the driving force with an object moving with the oscillator, this object will have transformed the available initial elastic energy into its own kinetic energy.

Resonance and critical damping

Earlier we saw the approximation for the average amplitude of the mechanical energy, if the damping factor were equal to zero. However if the damping factor depends on the angular frequency of the driving force, $\gamma=\gamma(\omega)$, then it cannot be zero.

A more interesting case arises if we suppose that the damping factor $\gamma(\omega)$ at resonance, $\omega=\omega_0$, is equal to $2\omega_0$, thus introducing to the system the condition of critical damping,

$$\omega'^2 = \omega_0^2 - \frac{\gamma_0^2}{4}$$

$$\gamma_0 = 2\omega_0 \Rightarrow$$

$$\omega'^2 = \omega_0^2 - \frac{4\omega_0^2}{4} = 0$$

where ω' is the angular frequency referring to the damping.

On the other hand, we have the condition which maximizes the amplitude $y_0(\omega)$ of the driven oscillator,

$$\omega^2 = \omega_0^2 - \frac{\gamma_0^2}{2}$$

Although the angular frequency ω of the driving force is generally considered that it does not depend on the angular frequency ω' of the damping, if we substitute $\gamma_0=2\omega_0$ in the latter expression, we take

$$\omega^2 = \omega_0^2 - \frac{\gamma_0^2}{2}$$

$$\gamma_0 = 2\omega_0 \Rightarrow$$

$$\omega^2 = \left|\omega_0^2 - \frac{4\omega_0^2}{2}\right| = |\omega_0^2 - 2\omega_0^2| = |-\omega_0^2| = \omega_0^2 = \frac{\gamma_0^2}{4} \Rightarrow$$

$$\omega = \omega_0 = \frac{\gamma_0}{2}$$

By taking the absolute value of the condition for the maximum amplitude $y_0(\omega)_{max}$, we may suggest that the angular frequency ω of the driving force can be equal to the angular frequency ω_0 of the

free oscillator from two opposite directions, either if $\gamma=0$ (before damping or a driving force is applied), or if $\gamma=2\omega_0$ (when damping and a driving force are applied).

Thus for the driven and damped harmonic oscillator, when we have critical damping, $\omega'=0$, we also have resonance, $\omega=\omega_0$.

Substituting now the condition of critical damping, $\gamma_0=2\omega_0$, into the formula for the amplitude $y_0(\omega)$ we take,

$$y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{\gamma_0^2\omega^2 + (\omega_0^2 - \omega^2)^2}},$$

$$\gamma_0 = 2\omega_0 \Rightarrow$$

$$y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{4\omega_0^2\omega^2 + (\omega_0^2 - \omega^2)^2}}$$

The expression in the square root is in fact a perfect square,

$$4\omega_0^2\omega^2 + (\omega_0^2 - \omega^2)^2 = 4\omega_0^2\omega^2 + \omega_0^4 + \omega^4 - 2\omega_0^2\omega^2 = 2\omega_0^2\omega^2 + \omega_0^4 + \omega^4 = (\omega_0^2 + \omega^2)^2$$

Therefore, for the amplitude $y_0(\omega)$ it will be

$$y_0(\omega) = \frac{F_0}{m}\frac{1}{\sqrt{4\omega_0^2\omega^2 + (\omega_0^2 - \omega^2)^2}} = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2 + \omega^2)^2}} = \frac{F_0}{m}\frac{1}{\omega_0^2 + \omega^2} = y_0\frac{\omega_0^2}{\omega_0^2 + \omega^2},$$

$$y_0 \equiv y_0(\omega = 0) = \frac{F_0}{m\omega_0^2} \Rightarrow$$

$$y_0(\omega) = y_0\frac{1}{1 + \frac{\omega^2}{\omega_0^2}}, \qquad \gamma_0 = 2\omega_0$$

so that, for the amplitude of the mechanical energy, we have

$$E_m(\omega) = E_{el}(\omega) + E_k(\omega) = \frac{1}{2}m\omega_0^2y_0^2(\omega) + \frac{1}{2}m\omega^2y_0^2(\omega) = \frac{1}{2}my_0^2(\omega)(\omega_0^2 + \omega^2) \Rightarrow$$

$$E_m(\omega) = \frac{1}{2}my_0^2\frac{\omega_0^2 + \omega^2}{\left(1 + \frac{\omega^2}{\omega_0^2}\right)^2} = \frac{1}{2}m\frac{F_0^2}{m^2\omega_0^4}\frac{\omega_0^2 + \omega^2}{\left(\frac{\omega_0^2 + \omega^2}{\omega_0^2}\right)^2}, \qquad y_0 = \frac{F_0}{m\omega_0^2} \Rightarrow$$

$$E_m(\omega) = \frac{F_0^2}{2m}\frac{1}{\omega_0^2 + \omega^2} \Rightarrow$$

$$E_m(\omega) = \frac{F_0^2}{2m}\frac{\omega_0^2}{\omega_0^2}\frac{1}{\omega_0^2 + \omega^2} = \frac{F_0^2}{2m\omega_0^2}\frac{\omega_0^2}{\omega_0^2 + \omega^2} = E_0\frac{1}{1 + \frac{\omega^2}{\omega_0^2}},$$

$$E_0 = \frac{F_0^2}{2m\omega_0^2}$$

If additionally $\omega=\omega_0$, then we have

$$E_m(\omega = \omega_0) = E_0\frac{1}{1 + \frac{\omega_0^2}{\omega_0^2}} = E_0\frac{1}{1+1} = \frac{1}{2}E_0$$

so that amplitude of the mechanical energy $E_m(\omega)$ will be half the total mechanical energy E_0.

Notes:

If in the expression which gives the amplitude

$$y_0(\omega) = y_0\frac{1}{1 + \frac{\omega^2}{\omega_0^2}}, \qquad \gamma_0 = 2\omega_0$$

we set

$$\omega = \kappa v, \qquad \omega_0 = \kappa c,$$

where κ is the wavenumber, then we take

$$y_0(v) = y_0\frac{1}{1 + \frac{v^2}{c^2}}$$

If in the last expression we add a factor of ½, and assume that that $v<<c$,

$$y_0(v) = y_0\frac{1}{1 + \frac{v^2}{2c^2}} = y_0\left(1 + \frac{v^2}{2c^2}\right)^{-1} \approx y_0\left(1 - \frac{v^2}{2c^2}\right) \approx y_0\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = \frac{1}{\gamma_L}y_0$$

where

$$\gamma_L = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}}$$

we take back the relativistic expression for the displacement.

The relativistic formula for the displacement can also be taken from the expression for the maximum amplitude of the driven and damped harmonic oscillator,

$$y_0(\omega)_{max} = y_0(\omega_0)\frac{1}{\sqrt{1-\frac{{\gamma_0}^2}{4\omega_0^2}}}$$

if we use the following identities,

$$\omega_0 y_0 \approx \frac{2\pi c}{\lambda_0}\frac{\lambda_0}{2\pi} = c, \qquad y_0 \approx \frac{1}{2\pi}\lambda_0,$$

$$v = \omega y_0,$$

$$v_0 = \omega_0 y_0 = \frac{\gamma_0}{2} y_0 = c,$$

$$\gamma_0 = 2\omega_0, \qquad \omega = \omega_0,$$

where we have used the cyclic approximation relating the linear displacement *y* to the wavelength *λ* of the oscillator, so that,

$$y_0(\omega)_{max} = y_0(\omega_0)\frac{1}{\sqrt{1-\frac{{\gamma_0}^2}{4\omega_0^2}}} = y_0(\omega_0)\frac{1}{\sqrt{1-\frac{{v_0}^2}{\omega_0^2 y_0^2}}} \approx y_0(\omega_0)\frac{1}{\sqrt{1-\frac{v_0^2}{c^2}}} \Rightarrow$$

$$y_0(\omega)_{max} = y_0(\omega_0)\gamma_L,$$

$$\gamma_L = \frac{1}{\sqrt{1-\frac{v_0^2}{c^2}}}$$

Such approximations justify earlier assumptions we made, as in the ship- in- the- sea example, where the Lorentz factor γ_L appeared with a positive sign,

$$\gamma_L^+ = \sqrt{\frac{1}{1+\frac{v^2}{c^2}}}$$

while the result we previously took

$$y_0(\omega) = y_0 \frac{1}{1+\frac{\omega^2}{\omega_0^2}}, \qquad \gamma_0 = 2\omega_0$$

$$y_0(v) = y_0 \frac{1}{1+\frac{\kappa^2 v^2}{\kappa^2 c^2}} = y_0 \frac{1}{1+\frac{v^2}{c^2}}, \qquad \omega = \kappa v, \qquad \omega_0 = \kappa c$$

may be considered more complete, as it will be further explored in what follows.

Exponential-sinusoidal driving force

Here an attempt will be made to find an expression for the solution of the driven and damped oscillator

$F(t) = F_0 \cos \omega t$

$ma + bv + ky = F_0 \cos \omega t$

$y(t) = C_1 e^{-\frac{\gamma}{2}t} \cos(\omega' t \pm \varphi') + C_2 \cos(\omega t \pm \varphi)$

$$C_1 = \frac{F_0}{m} \frac{\omega_0^2 - \omega^2}{\gamma^2 \omega^2 + (\omega_0^2 - \omega^2)^2}, \qquad C_2 \equiv y_0(\omega) = \frac{F_0}{m} \frac{1}{\sqrt{\gamma^2 \omega^2 + (\omega_0^2 - \omega^2)^2}}$$

which will be composed of one term (instead of two).

In order to do this we may suppose that the driving force also includes an exponential term,

$F(t) = F_0 e^{-\gamma t} \cos \omega t$

so that we take a solution in the following compact form

$y(t) = y_0(\omega) e^{-\gamma t} \cos(\omega t - \varphi)$

If we substitute this solution into the equation of motion

$ma + ky + bv = F_0 e^{-\gamma t} \cos \omega t$

we will in fact find the same expressions for the amplitude $y_0(\omega)$ and phase φ.

We have already done such a calculation. Here we will see how this calculation changes if we suppose that the driving force includes an exponential term, so that we have an equation of motion of the form,

$ma + ky + bv = F_0 e^{-\gamma t} \cos \omega t$

with the following solution,

$y(t) = y_0(\omega) e^{-\gamma t} \cos(\omega t - \varphi)$

$$v(t) = \frac{d}{dt}[y_0(\omega) e^{-\gamma t} \cos(\omega t - \varphi)] = y_0(\omega) \frac{d}{dt}[e^{-\gamma t} \cos(\omega t - \varphi)] \Rightarrow$$

$v(t) = -\gamma y_0(\omega)e^{-\gamma t}\cos(\omega t - \varphi) - \omega y_0(\omega)e^{-\gamma t}\sin(\omega t - \varphi)$

$a(t) = y_0(\omega)\dfrac{d}{dt}[-\gamma e^{-\gamma t}\cos(\omega t - \varphi) - \omega e^{-\gamma t}\sin(\omega t - \varphi)] \Rightarrow$

$a(t) = y_0(\omega)[\gamma^2 e^{-\gamma t}\cos(\omega t - \varphi) + \omega\gamma e^{-\gamma t}\sin(\omega t - \varphi)$

$+\omega\gamma e^{-\gamma t}\sin(\omega t - \varphi) - \omega^2 e^{-\gamma t}\cos(\omega t - \varphi)] \Rightarrow$

$a(t) = \gamma^2 y_0(\omega)e^{-\gamma t}\cos(\omega t - \varphi) + 2\omega\gamma y_0(\omega)e^{-\gamma t}\sin(\omega t - \varphi) - \omega^2 y_0(\omega)e^{-\gamma t}\cos(\omega t - \varphi)$

Substituting this solution into the equation of motion of the driven and damped harmonic oscillator, we take

$ma + ky + bv = F_0 \cos\omega t\, e^{-\gamma t} \Rightarrow$

$m\gamma^2 y_0(\omega)e^{-\gamma t}\cos(\omega t - \varphi) + 2m\omega\gamma y_0(\omega)e^{-\gamma t}\sin(\omega t - \varphi)$

$-m\omega^2 y_0(\omega)e^{-\gamma t}\cos(\omega t - \varphi) + k y_0(\omega)e^{-\gamma t}\cos(\omega t - \varphi)$

$-b\gamma y_0(\omega)e^{-\gamma t}\cos(\omega t - \varphi) - b\omega y_0(\omega)e^{-\gamma t}\sin(\omega t - \varphi) =$

$F_0 e^{-\gamma t}\cos\omega t$

The exponential term disappears from both sides of the previous equation, so that

$m\gamma^2 y_0(\omega)\cos(\omega t - \varphi) + 2m\omega\gamma y_0(\omega)\sin(\omega t - \varphi)$

$-m\omega^2 y_0(\omega)\cos(\omega t - \varphi) + k y_0(\omega)\cos(\omega t - \varphi)$

$-b\gamma y_0(\omega)\cos(\omega t - \varphi) - b\omega y_0(\omega)\sin(\omega t - \varphi) =$

$F_0 \cos\omega t \Rightarrow$

$m\omega\gamma y_0(\omega)\sin(\omega t - \varphi) - m\omega^2 y_0(\omega)\cos(\omega t - \varphi) + m\omega_0^2 y_0(\omega)\cos(\omega t - \varphi) =$

$F_0 \cos\omega t$

This expression is identical to the expression we previously took, without the exponential term.

Thus the exponential- sinusoidal solution

$ma + bv + ky = F_0 e^{-\gamma t}\cos\omega t$

$F(t) = F_0 e^{-\gamma t}\cos\omega t$

$y(t) = y_0(\omega)e^{-\gamma t}\cos(\omega t - \varphi)$

gives the same expressions for the phase φ and the amplitude $y_0(\omega)$, with those given by the common steady state solution.

However the exponential- sinusoidal solution has the advantage of 'remembering' the initial conditions, because it includes the damping factor in the exponential term. Still it is a solution of the driven damped oscillator, because the amplitude and the phase depend on the driving angular frequency ω.

Letting, for example, the time evolve, $t>>1$, the exponential term of the solution vanishes, and the solution becomes that of the damped oscillator (without a driving force),

$ma + bv + ky = F_0 e^{-\gamma t} \cos \omega t$

$t \gg 1, \qquad e^{-\gamma t} \to 0 \Rightarrow$

$ma + bv + ky = 0$

On the other hand, supposing a small damping factor, $\gamma \approx 0$, we take back the standard (steady state) solution of the forced damped oscillator (without the exponential term),

$ma + bv + ky = F_0 e^{-\gamma t} \cos \omega t$

$\gamma \approx 0, \qquad e^{-\gamma t} \approx 1 \Rightarrow$

$ma + bv + ky = F_0 \cos \omega t$

Thus we have a solution which oversees the special cases.

Additionally the assumption of an exponentially- sinusoidal force is not unrealistic, taking into account that the periodic force which is imposed by the moving object onto the oscillating system will not be applied forever, so that it will decrease and vanish in due time.

Here we will consider the exponential- sinusoidal solution, if the frequency ω of the driving force is sufficiently larger than the natural frequency ω_0 of the oscillator.

For simplicity we may set the phase equal to zero, $\varphi=0$,

$$|\tan\varphi| = \left|\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right| \approx \frac{\omega}{\omega^2} = \frac{1}{\omega} \approx 0 \Rightarrow$$

$$\varphi = 0, \qquad \omega \gg \omega_0$$

Also the expression for the speed can be simplified as follows,

$$v(t) = -\gamma y_0(\omega)e^{-\gamma t}\cos(\omega t - \varphi) - \omega y_0(\omega)e^{-\gamma t}\sin(\omega t - \varphi) \Rightarrow$$

$$v(t) = -\gamma y_0(\omega)e^{-\gamma t}\cos\omega t - \omega y_0(\omega)e^{-\gamma t}\sin\omega t\,, \qquad \varphi = 0 \Rightarrow$$

$$v(t) \approx -\omega y_0(\omega)e^{-\gamma t}\sin\omega t\,, \qquad \omega \gg \gamma$$

where we have supposed that here the damping factor γ refers to γ_0.

Considering now the energy corresponding to the exponential- sinusoidal solution for large ω, $\omega >> \omega_0$,

$$ma + bv + ky = F_0 e^{-\gamma t}\cos\omega t$$

$$F(t) = F_0 e^{-\gamma t}\cos\omega t$$

$$y(t) = y_0(\omega)e^{-\gamma t}\cos\omega t$$

$$v(t) \approx -\omega y_0(\omega)e^{-\gamma t}\sin\omega t$$

we have for the mechanical energy which we may simply call $E(t)$, that,

$$E(t) = E_{el}(t) + E_k(t) = \frac{1}{2}ky^2(t) + \frac{1}{2}mv^2(t) \Rightarrow$$

$$E(t) \approx \frac{1}{2}m\omega_0^2 y_0^2(\omega)e^{-2\gamma t}\cos^2\omega t + \frac{1}{2}m\omega^2 y_0^2(\omega)e^{-2\gamma t}\sin^2\omega t \Rightarrow$$

$$E(t) \approx \frac{1}{2}m\omega^2 y_0^2(\omega)e^{-2\gamma t}\sin^2\omega t = E_k(t), \qquad \omega > \omega_0$$

Thus, if $\omega > \omega_0$, the mechanical energy is mostly kinetic energy.

The last expression can also be written as follows,

$$E(t) \approx \frac{1}{2}m\omega^2 y_0^2(\omega)e^{-2\gamma t}\sin^2\omega t\,, \qquad \omega > \omega_0 \Rightarrow$$

$$E(t) = \frac{F_0^2}{2m\gamma^2(\omega)}e^{-2\gamma_0 t}\sin^2\omega t\,, \qquad y_0(\omega) = \frac{F_0}{m\omega\gamma(\omega)} \Rightarrow$$

$$E(t) = E(\omega)e^{-2\gamma_0 t}\sin^2\omega t\,, \qquad E(\omega) = \frac{F_0^2}{2m\gamma^2(\omega)}$$

Some caution is needed with respect to the distinction between the variable damping factor $\gamma(\omega)$, and the constant damping factor $\gamma \equiv \gamma_0$.

Here is the expanded form of the previous equation,

$$E(t) = E(\omega)e^{-2\gamma_0 t}\sin^2\omega t \Rightarrow$$

$$E(t) = \frac{F_0^2}{2m\gamma_0^2}\frac{1}{1+\frac{1}{\gamma_0^2}\frac{(\omega_0^2-\omega^2)^2}{\omega^2}}e^{-2\gamma_0 t}\sin^2\omega t,$$

$$E(\omega) = \frac{F_0^2}{2m\gamma^2(\omega)} = \frac{F_0^2}{2m\gamma_0^2}\frac{1}{1+\frac{1}{\gamma_0^2}\frac{(\omega_0^2-\omega^2)^2}{\omega^2}},$$

$$\gamma(\omega) = \frac{F_0}{m\omega y_0(\omega)} = \gamma_0\sqrt{1+\frac{1}{\gamma_0^2}\frac{(\omega_0^2-\omega^2)^2}{\omega^2}}, \qquad \gamma_0 = \frac{F_0}{m\omega_0 y_0(\omega_0)}$$

The energy

$$E(t) = E(\omega)e^{-2\gamma_0 t}\sin^2\omega t$$

can be treated either with respect to the time t, keeping the driving angular frequency ω fixed, or in relation to the frequency ω, keeping the time t fixed, including thus both the aspects of damping, and of a driving force, in a single expression.

Comparison with relativistic energy

Here we will make a comparison between the formulas for the amplitude and the energy of the driven and damped harmonic oscillator, and the respective formulas in relativity.

One may wonder what the relationship could be between an object moving in spacetime, and an object oscillating 'on a spring.' Nevertheless, we have already seen in previous examples (the ship-in- the- sea, and the rocket- in- space, or even the spaceship- on- the- brachistochrone) that if spacetime is an oscillating medium, then the motion of an object can be approximated by the equations of an oscillator. The connection between the oscillator and the brachistochrone will be established later on.

The new element here is that, besides the harmonic and the damping force of the damped harmonic oscillator, we also have the driving force. This force can either be seen as independent from the other forces, imposed by an external source onto the oscillator, or, better, as a force intimately related to the other forces, caused by the moving object itself (by the engine of a spaceship for example). Consequently, in the latter sense, the motion of the object can be described by the equation of the driven and damped harmonic oscillator.

As far as the displacement of the moving object is concerned, instead of regarding a vertical displacement $y(t)$, we may equivalently consider a horizontal displacement $x(t)$. But the description of the problem will be the same. As long as the moving object can be seen as a mass 'hanging from a spring' (where in this case the spring is a wave in spacetime), the amplitude of the oscillation $y(t)=y_0(\omega)cos(\omega t)$ will give the displacement of the object.

Now will make some considerations with respect to the speed of the object. The speed is taken from the equation of the oscillator, and its steady state solution,

$$ma + bv + ky = F_0 \cos \omega t$$

$$y(t) = y_0(\omega) \cos(\omega t - \varphi)$$

$$v(t) = \frac{dy(t)}{dt} = -\omega y_0(\omega) \sin(\omega t - \varphi) = -v_0(\omega) \sin(\omega t - \varphi),$$

where

$$v_0(\omega) = \omega y_0(\omega)$$

is the amplitude of the speed *v(t)*.

The speed *v(t)* takes its maximum value, at $\omega=\omega_0$, and at some time $t=T_0$. To see this, using the boundary condition

$$t = t_0 = T_0, \qquad \omega = \omega_0, \qquad \varphi = \frac{\pi}{2}, \qquad T = \frac{2\pi}{\omega} \equiv \frac{2\pi}{\omega_0} \equiv T_0$$

$$\sin(\omega t - \varphi) = \sin\left(\frac{2\pi}{T_0}T_0 - \frac{\pi}{2}\right) = \sin\left(2\pi - \frac{\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1,$$

$$\cos(\omega t - \varphi) = \cos\left(\frac{2\pi}{T_0}T_0 - \frac{\pi}{2}\right) = \cos\left(2\pi - \frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0$$

$$\cos\omega t = \cos\frac{2\pi}{T_0}T_0 = \cos 2\pi = 1,$$

$$y(t = T_0) = 0, \qquad v(t = T_0) = \omega_0 y_0(\omega_0) = v_0(\omega_0), \qquad a(t = T_0) = 0,$$

$$F_0 = F(t = T_0) = b v_0(\omega_0),$$

and the values for the amplitude $y_0(\omega)$ at $\omega=0$, and $\omega=\omega_0$,

$$y_0(\omega = 0) \equiv y_0 = \frac{F_0}{m\omega_0^2}$$

$$y_0(\omega = \omega_0) = \frac{F_0}{m\gamma_0\omega_0} = \frac{F_0}{b_0\omega_0}, \qquad \gamma(\omega) = \frac{b(\omega)}{m}, \qquad \gamma(\omega_0) = \frac{b(\omega_0)}{m} \equiv \gamma_0,$$

where

$$F_0 = m\omega_0^2 y_0 = m\gamma_0\omega_0 y_0(\omega_0) \Rightarrow$$

$$\omega_0 y_0 = \gamma_0 y_0(\omega_0)$$

we have that,

$$v(t) = -v_0(\omega)\sin(\omega t - \varphi) \Rightarrow$$

$$v_{max} = v_0(\omega_0) = \omega_0 y_0(\omega_0) \Rightarrow$$

$$v_{max} = \omega_0 y_0(\omega_0) = \omega_0 \frac{F_0}{m\gamma_0\omega_0} = \frac{F_0}{m\gamma_0} = \frac{F_0}{b_0} \Rightarrow$$

$$v_{max} = \frac{F_0}{b_0} = \frac{y_0 m\omega_0^2}{m\gamma_0} = \frac{y_0\omega_0^2}{\gamma_0} \approx \frac{y_0\omega_0^2}{\omega_0} = y_0\omega_0, \qquad \gamma_0 \approx \omega_0,$$

$$y_0\omega_0 \approx \frac{\lambda_0}{2\pi}\frac{2\pi c}{\lambda_0} = c \;\Rightarrow$$

$$v_{max} = v_0(\omega_0) = \omega_0 y_0(\omega_0) \approx \gamma_0 y_0(\omega_0) = \omega_0 y_0 \approx c$$

Thus the speed

$$v(t) = -v_0(\omega)\sin(\omega t - \varphi),$$

$$v_0(\omega) = \omega y_0(\omega)$$

can be no greater than the speed of light c (presumably the speed of the propagating wave).

Alternatively, we can define a speed of the following form,

$$v = \omega y_0$$

The perspective of this speed is that it can be greater than the speed of light c,

$$v = \omega y_0 = n\omega_0 y_0 \approx n\frac{2\pi c}{\lambda_0}\frac{\lambda_0}{2\pi} = nc, \qquad y_0 = \frac{\lambda_0}{2\pi}, \qquad n = \frac{\omega}{\omega_0},$$

$$v_0 = \omega_0 y_0 = c, \qquad n = 1$$

The number n which appears here refers in fact to the harmonics of the oscillator,

$$n = \frac{\omega_n}{\omega_0} = \frac{2\pi}{T_n}\frac{T_0}{2\pi} = \frac{T_0}{T_n} = \frac{\lambda_0}{c}\frac{c}{\lambda_n} = \frac{\lambda_0}{\lambda_n}$$

Thus the speed

$$v_n = \omega_n y_0 = n\omega_0 y_0 \approx nc$$

can be directly linked to all our previous considerations about motion in spacetime, and motion on the brachistochrone. Such motion will be further described later on.

A comparison between the speed v_n, and the speed amplitude $v_0(\omega)$, is the following one,

$$\frac{v_0(\omega)}{v_n} = \frac{\omega y_0(\omega)}{\omega y_0} = \frac{y_0(\omega)}{y_0} = \frac{F_0}{m\omega\gamma(\omega)}\frac{m\omega_0^2}{F_0} = \frac{\omega_0^2}{\omega\gamma(\omega)},$$

$$y_0(\omega) = \frac{F_0}{m\omega\gamma(\omega)}, \qquad y_0 \equiv y_0(0) = \frac{F_0}{m\omega_0^2} \Rightarrow$$

$$\frac{v_0(\omega)}{v_n} \approx \frac{\omega_0^2}{\omega_n^2} = \frac{1}{n^2}, \qquad n > 1, \qquad \gamma(\omega_n) \approx \omega_n,$$

where, setting $n=1$, $v_n=v_0$, we have

$$\frac{v_0(\omega)}{v_n} = \frac{v_0(\omega_0)}{v_0} = \frac{\omega_0 y_0(\omega_0)}{\omega_0 y_0} = \frac{y_0(\omega_0)}{y_0} = \frac{\omega_0}{\gamma_0} \approx 1, \qquad \omega_0 \approx \gamma_0$$

so that, specifically, for the speed v_n we have that,

$$\frac{v_0}{v_n} = \frac{\omega_0 y_0}{\omega_n y_0} = \frac{\omega_0}{\omega_n} = \frac{1}{n}, \qquad \omega_n = n\omega_0 \Rightarrow$$

$$v_n = nv_0 = nc, \qquad v_0 = c$$

After defining such a speed, in order to make the comparison with the relativistic expressions for the displacement and the energy, we need first to define the damping factor $\gamma(\omega)$ with respect to the speed v_n, where

$$\gamma(\omega) = \gamma_0\sqrt{1 + \frac{1}{\gamma_0^2}\frac{(\omega_0^2 - \omega^2)^2}{\omega^2}}$$

To do this, we have to manipulate the fraction, with the angular frequencies ω and ω_0, which appears in the square root of the previous expression.

Introducing the wavenumber,

$$\kappa = \frac{2\pi}{\lambda_0} = \frac{2\pi}{cT_0} = \frac{\omega_0}{c}$$

we may express the angular frequencies ω and ω_0 with respect to the speeds v and c, respectively, as follows,

$$\omega_0 = \kappa c$$

$$\omega_n = n\omega_0 = n\kappa c = \kappa v_n,$$

$$v_n = nc, \qquad v_0 = c, \qquad n = 1$$

Thus the quantity which appears in the expression for the damping factor $\gamma(\omega)$, takes the form,

$$\frac{\left(\omega_0^2 - \omega^2\right)^2}{\omega^2} = \frac{\left(\kappa^2 c^2 - \kappa^2 v^2\right)^2}{\kappa^2 v^2} = \kappa^2 \frac{\left(c^2 - v^2\right)^2}{v^2}$$

where here the speed v refers to the speed v_n, and the damping factor γ takes the form

$$\gamma(\omega) = \gamma_0 \sqrt{1 + \frac{1}{\gamma_0^2} \frac{(\omega_0^2 - \omega^2)^2}{\omega^2}} \rightarrow$$

$$\gamma(v) = \gamma_0 \sqrt{1 + \frac{\kappa^2}{\gamma_0^2} \frac{\left(c^2 - v^2\right)^2}{v^2}}$$

Now, in order to find a corresponding expression for the amplitude of the displacement $y_0(\omega)$, with respect to the speed v, from the following formulas

$$y_0(\omega) = \frac{F_0}{m} \frac{1}{\sqrt{\gamma_0^2 \omega^2 + (\omega_0^2 - \omega^2)^2}},$$

$$y_0 = y_0(\omega = 0) = \frac{F_0}{m\omega_0^2}, \qquad y_0(\omega = \omega_0) = \frac{F_0}{m\gamma_0\omega_0},$$

$$y_0(\omega) = \frac{F_0}{m\gamma(\omega)\omega} = y_0(0)\frac{\omega_0^2}{\gamma(\omega)\omega} = y_0(\omega_0)\frac{\gamma_0\omega_0}{\gamma(\omega)\omega}, \qquad \frac{y_0(0)}{y_0(\omega_0)} = \frac{\gamma_0}{\omega_0}$$

$$\gamma(\omega) = \gamma_0 \sqrt{1 + \frac{1}{\gamma_0^2} \frac{(\omega_0^2 - \omega^2)^2}{\omega^2}}, \qquad \gamma(v) = \gamma_0 \sqrt{1 + \frac{\kappa^2}{\gamma_0^2} \frac{\left(c^2 - v^2\right)^2}{v^2}}$$

and substituting

$$\omega_0 = \kappa c, \qquad \omega = \kappa v$$

we have

$$y_0(\omega) = \frac{F_0}{m} \frac{1}{\sqrt{\gamma_0^2 \omega^2 + (\omega_0^2 - \omega^2)^2}} = \frac{F_0}{m\gamma(\omega)\omega} = y_0(0)\frac{\omega_0^2}{\gamma(\omega)\omega} = y_0(\omega_0)\frac{\gamma_0\omega_0}{\gamma(\omega)\omega} \rightarrow$$

$$y_0(v) = \frac{F_0}{m}\frac{1}{\sqrt{\gamma_0^2\kappa^2v^2 + \kappa^4(c^2 - v^2)^2}} = \frac{F_0}{m\kappa}\frac{1}{\sqrt{\gamma_0^2v^2 + \kappa^2(c^2 - v^2)^2}},$$

$$y_0(v) = \frac{F_0}{m\kappa}\frac{1}{\gamma(v)v} = y_0(0)\frac{\kappa c^2}{\gamma(v)v} = y_0(\omega_0)\frac{\gamma_0 c}{\gamma(v)v}, \qquad \frac{y_0(0)}{y_0(\omega_0)} = \frac{\gamma_0}{\kappa c}$$

Before plotting the graph for the amplitude of the displacement $y_0(v)$, we will also find the corresponding expression for the energy amplitude with respect to the speed v, and then make any assumptions necessary in order to simplify the expressions.

The energy amplitude $E_m(\omega)$, or simply $E(\omega)$, is given, as we have already seen, by the following formula,

$$E(\omega) = \frac{F_0^2}{2m\gamma^2(\omega)}\left(1 + \frac{\omega_0^2}{\omega^2}\right) = E_0\frac{\omega_0^2}{\gamma^2(\omega)}\left(1 + \frac{\omega_0^2}{\omega^2}\right),$$

$$E_0 = \frac{F_0^2}{2m\omega_0^2},$$

The same amplitude can be expressed with respect to the speed v, as follows

$$E(v) = E_0\frac{\kappa^2c^2}{\gamma^2(v)}\left(1 + \frac{\kappa^2c^2}{\kappa^2v^2}\right) = E_0\frac{\kappa^2c^2}{\gamma^2(v)}\left(1 + \frac{c^2}{v^2}\right),$$

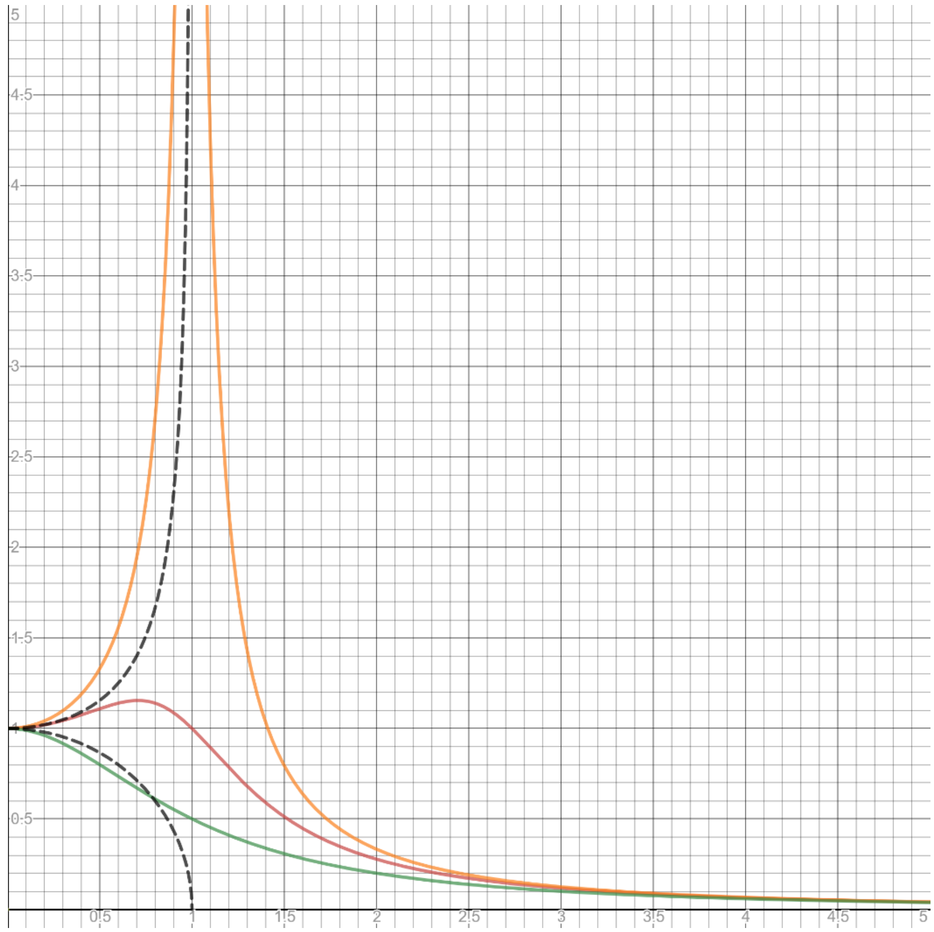
$$\omega = \kappa v, \qquad \omega_0 = \kappa c,$$

and in an expanded form as

$$E(v) = E_0\frac{\kappa^2c^2}{\gamma^2(v)}\left(1 + \frac{c^2}{v^2}\right) = E_0\frac{\kappa^2c^2}{\gamma_0^2\left[1 + \frac{\kappa^2}{\gamma_0^2}\frac{(c^2 - v^2)^2}{v^2}\right]}\left(1 + \frac{c^2}{v^2}\right),$$

$$\gamma(v) = \gamma_0\sqrt{1 + \frac{\kappa^2}{\gamma_0^2}\frac{(c^2 - v^2)^2}{v^2}}$$

Now we are able to plot the following couple of graphs. The first one refers to the amplitude of the displacement $y_0(v)$:



The displacement $y_0(v)$ of an object as a function of its speed v

The function depicted is,

$$y_0(v) = y_0 \frac{\kappa c^2}{\gamma(v) v} = y_0 \frac{\kappa c^2}{\sqrt{\gamma_0^2 v^2 + \kappa^2 (c^2 - v^2)^2}} \approx \frac{1}{\sqrt{\gamma_0^2 v^2 + (1 - v^2)^2}},$$

where

$$\gamma(v) = \gamma_0 \sqrt{1 + \frac{\kappa^2}{\gamma_0^2} \frac{\left(c^2 - v^2\right)^2}{v^2}} = \sqrt{\gamma_0^2 + \kappa^2 \frac{\left(c^2 - v^2\right)^2}{v^2}} \approx \sqrt{\gamma_0^2 + \frac{\left(1 - v^2\right)^2}{v^2}}\ ,\ \ \kappa = c = 1$$

The previous graph is similar to the one we plotted earlier for the amplitude $y_0(\omega)$, where here the orange line corresponds to a small damping factor ($\gamma_0=0.1$), the green line corresponds to critical damping ($\gamma_0=2\omega_0$, $\omega_0=1$), while the red line corresponds to a damping factor equal to one ($\gamma_0=1$).

The difference here is that we have included the relativistic expressions for the displacement of the moving object (black dotted lines),

$$y(v) = \gamma_L y_0 = y_0 \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \approx y_0 \frac{1}{\sqrt{1 - v^2}}, \qquad c = 1$$

$$\gamma_L \to \infty, \qquad v = c = 1$$

or

$$y(v) = \frac{1}{\gamma_L} y_0 = y_0 \sqrt{1 - \frac{v^2}{c^2}} \approx y_0 \sqrt{1 - v^2}, \qquad c = 1$$

$$\frac{1}{\gamma_L} \to 0, \qquad v = c = 1$$

where

$$\gamma_L = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \approx \frac{1}{\sqrt{1 - v^2}}, \qquad c = 1$$

is the relativistic Lorentz factor.

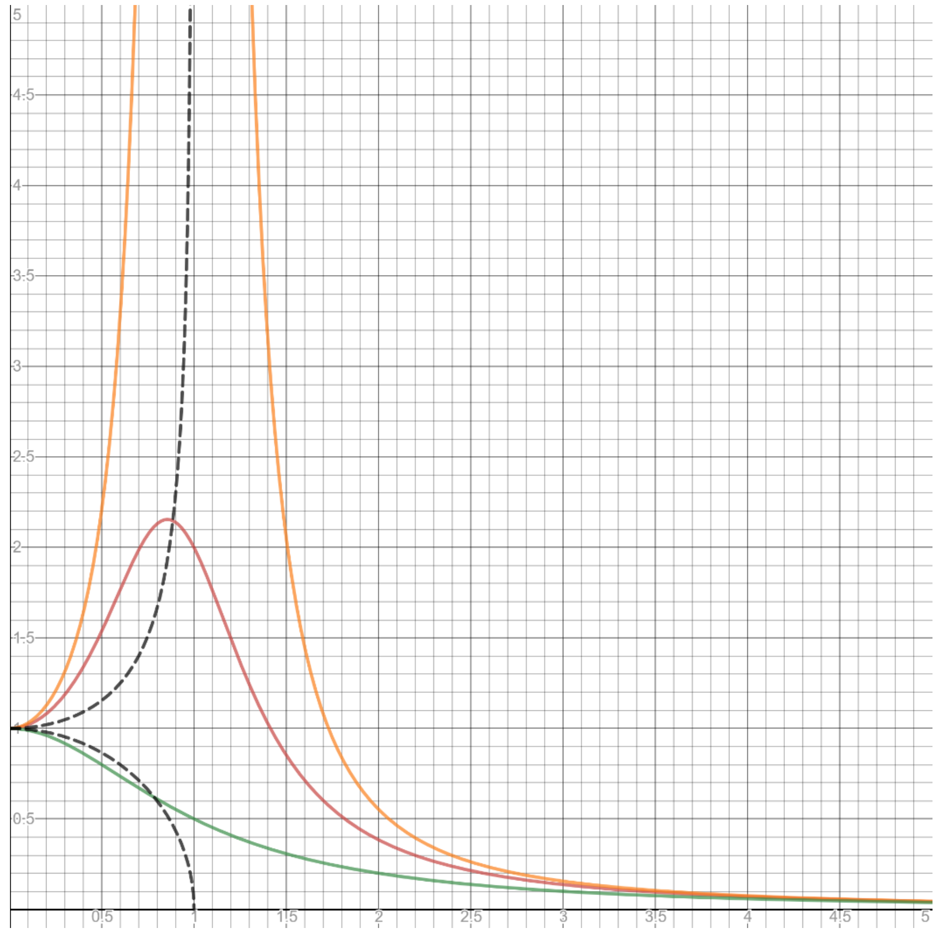
The reason why I have included both relativistic expressions, is because it is rather ambiguous which distance is ‘contracted’ or ‘expanded’ in relativity.

For example, supposing that the displacement (the amplitude) is expanded with respect to a moving observer, then we have to compare the upward black dotted line (corresponding to γ_L) to the orange line in the previous graph (corresponding to a small damping factor, $\gamma_0=0.1$, $\gamma_0^2 \approx 0$). But, as the speed of the moving object v reaches the speed c of the oscillator, while the amplitude according to relativity will go to infinity, the amplitude can still be defined according to the orange line, even if $v>c$.

If we compare the upward black dotted line, to the red line (corresponding to a damping factor equal to one ($\gamma_0 \approx \omega_0 \approx 1$), then these two lines begin to diverge somewhere at $v=0.5c$. But, while again the amplitude for the moving observer, according to relativity, will go to infinity, if $v=c$, the amplitude $y_0(v)$ will be equal to the original amplitude y_0, at $v=c$, according to the red line, while it can still be defined, if $v>c$.

On the other hand, if we suppose that the distance is contracted with respect to a moving observer, then we may compare the downward black dotted line (corresponding to $1/\gamma_L$), to the green line in the previous graph (corresponding to critical damping, $\gamma_0=2\omega_0 \approx 2$, $\omega_0=1$). These two lines begin to diverge somewhere at $v=0.7c$. But, while, in this case, the displacement according to relativity will go to zero, as the speed of the moving object v reaches the speed of light c, the displacement can still be defined according to the green line, even if $v>c$.

Similar considerations can be made with respect to the amplitude of the energy $E(v)$, according to the following second graph:



The energy amplitude *E(v)* as a function of the speed *v*

The function depicted is

$$E(v) = E_0 \frac{\kappa^2 c^2}{\gamma^2(v)}\left(1 + \frac{c^2}{v^2}\right) = E_0 \frac{\kappa^2 c^2}{\gamma_0^2 + \kappa^2 \frac{\left(c^2 - v^2\right)^2}{v^2}}\left(1 + \frac{c^2}{v^2}\right) \Rightarrow$$

$$E(v) \approx E_0 \frac{1}{\gamma_0^2 + \frac{\left(1 - v^2\right)^2}{v^2}}\left(1 + \frac{1}{v^2}\right)$$

where, again,

$$\gamma(v) = \gamma_0 \sqrt{1 + \frac{\kappa^2}{\gamma_0^2} \frac{\left(c^2 - v^2\right)^2}{v^2}} = \sqrt{\gamma_0^2 + \kappa^2 \frac{\left(c^2 - v^2\right)^2}{v^2}} \approx \sqrt{\gamma_0^2 + \frac{\left(1 - v^2\right)^2}{v^2}}\ , \ \ \kappa = c = 1$$

The red, orange, and green lines correspond to a damping factor $\gamma_0=1$, $\gamma_0=0.1$ ($\gamma_0^2\approx 0$), and $\gamma_0=2\omega_0=2$, $\omega_0=1$, respectively.

The relativistic expression for the energy is given by the following formula,

$$E(v) = \frac{1}{\gamma_L} E_{Ein} \Rightarrow$$

$$E_{Ein} = \gamma_L E(v)$$

This notation suggests that the total energy is E_{Ein}, because the Lorentz factor γ_L increases as the speed v of the object increases, while the total energy will go to infinity if the speed v of the object reaches the speed of light c,

$$\gamma_L = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx \frac{1}{\sqrt{1-v^2}}, \qquad c = 1$$

$$v = c = 1 \Rightarrow$$

$$\gamma_L = \frac{1}{\sqrt{1-v^2}} = \frac{1}{0} \to \infty \Rightarrow$$

$$E_{Ein} = \gamma_L E(v) \to \infty, \qquad v = c = 1$$

(black dotted lines in the previous graph).

On the other hand, we have the energy amplitude $E_0(\omega)$, or $E_0(v)$, as it was defined here,

$$E(\omega) = \frac{F_0^2}{2m\gamma^2(\omega)}\left(1+\frac{\omega_0^2}{\omega^2}\right) = E_0 \frac{\omega_0^2}{\gamma^2(\omega)}\left(1+\frac{\omega_0^2}{\omega^2}\right),$$

$$E(v) = E_0 \frac{\kappa^2 c^2}{\gamma^2(v)}\left(1+\frac{c^2}{v^2}\right),$$

$$\omega = \kappa v, \qquad \omega_0 = \kappa c,$$

$$E_0 = \frac{F_0^2}{2m\omega_0^2} = \frac{F_0^2}{2m\kappa^2 c^2}$$

Supposing that the speed of the moving object v is larger than the speed of light c, we have that

$$\omega > \omega_0, \qquad \gamma(\omega) \approx \omega,$$

$$v > c, \qquad \gamma(v) \approx v \Rightarrow$$

$$E(v) = E_0 \frac{\kappa^2 c^2}{\gamma^2(v)}\left(1 + \frac{c^2}{v^2}\right) \approx E_0 \frac{\kappa^2 c^2}{\gamma^2(v)} \approx E_0 \frac{\kappa^2 c^2}{v^2} \approx \frac{1}{v^2} E_0, \qquad \kappa c = 1$$

so that the energy (which in fact at large speeds is purely kinetic energy) of the object can be defined for all speeds, $v>>c$, as it goes to zero, $E(v) \to 0$, only if the speed of the object goes to infinity, $v \to \infty$.

On the other hand, at small speeds, $v<<c$, the energy amplitude $E(v)$ will approximately be equal to the initial energy E_0 (which is mostly elastic energy at small speeds).

If the speed is close to the speed of light, $v \approx c$, which is the case of critical damping, $\gamma_0 = 2\omega_0 = 2\kappa c$, as we have already seen it will be,

$$E(\omega) = E_0 \frac{1}{1 + \frac{\omega^2}{\omega_0^2}}, \qquad E_0 = \frac{F_0^2}{2m\omega_0^2},$$

$$E(v) = E_0 \frac{1}{1 + \frac{v^2}{c^2}} = \frac{1}{2} E_0, \qquad v = c,$$

$$\omega = \kappa v, \qquad \omega_0 = \kappa c,$$

$$E_0 = \frac{F_0^2}{2m\omega_0^2} = \frac{F_0^2}{2m\kappa^2 c^2}$$

Thus at the speed of light the moving object utilizes half the available energy.

If we assume a small damping factor, then the energy amplitude will go to infinity, if the speed of the object approaches the speed of light. This would be the consequence of ignoring the medium (spacetime), in which objects move. By acknowledging the existence of the medium, and supposing that it oscillates, then we may use the equations of an oscillator in order to describe the motion of the object. Such equations give us for the displacement and the energy of the object formulas which can be defined even if the speed of the object exceeds the speed of light. Later on we will establish the connection between these equations and motion on the brachistochrone.

Inertial mass- gravitational charge equivalence

Throughout this discussion, we have supposed that the acceleration, which we may call a, gained by an object of mass m moving on the brachistochrone, can be identified with the gravitational acceleration g due to the mass M_B of the brachistochrone.

Such an assumption is based on the inertial mass- gravitational charge equivalence. That is, if m is the inertial mass of the object, and q is the object's gravitational charge, then it will be

$$ma = qg$$

In all experiments of free fall it has been measured that the acceleration a of a free falling object is always equal to the acceleration g of the gravitational field, so that the object's inertial mass m will always be identified with its gravitational charge q,

$$ma = qg,$$

$$a \equiv g \Rightarrow$$

$$m \equiv q$$

However this may not be the general case. For example, in the same experiments it has been assumed that the air's resistance is negligible. This force of resistance is assumed to be proportional to the object's speed v, where the constant of proportionality is the damping constant b. Including such a force of resistance, we will have an equation of motion of the form

$$ma + bv = qg$$

where

$$a = \frac{q}{m}g - \frac{b}{m}v,$$

$$m = q\frac{g}{a} - b\frac{v}{a}$$

If we suppose that the acceleration a of the moving object is equal to the gravitational acceleration g, then it will be

$$m = q\frac{g}{a} - b\frac{v}{a},$$

$$a = g \Rightarrow$$

$$m = q - b\frac{v}{a}$$

The object's inertial mass m will be equal to its gravitational charge only if the resistance $b=0$ is zero, so that the objet doesn't move, $v=0$.

On the other hand, if we assume that the inertial mass m of the object is always equal to its gravitational charge q, then it will be

$$a = \frac{q}{m}g - \frac{b}{m}v$$

$$m = q \Rightarrow$$

$$a = g - \frac{b}{m}v$$

so that the object's acceleration a will be equal to the gravitational acceleration g only if damping is again not present, $b=0$, $v=0$.

Thus, as long as the object is moving with some speed v, and damping is present, $b \neq 0$, the related parameters will be different, $a \neq g$, $m \neq q$.

What is going on can be seen by acknowledging the following boundary conditions (assuming that the acceleration g of the gravitational field is always present, and constant at least at the boundaries),

$$t = 0, \qquad a = a_0, \qquad v = 0, \qquad g = g_0 \Rightarrow$$

$$ma_0 = qg_0,$$

and

$$t = t_0, \qquad a = 0, \qquad v = v_0, \qquad g = g_0 \Rightarrow$$

$$bv_0 = qg_0,$$

so that

$$bv_0 = ma_0 \Rightarrow$$
$$\gamma = \frac{b}{m} = \frac{a_0}{v_0} = \frac{1}{t_0}$$

where, according to these boundary conditions, we recover the damping factor γ.

The meaning of this is that the inertial mass- gravitational charge, or inertial acceleration- gravitational acceleration, equivalence, $m \equiv q$, $a \equiv g$, will be true on the boundaries. In other words, inbetween the boundaries, some of the properties of the system will be *hidden*.

In order to show this, supposing that the acceleration of gravity is constant, $g \equiv g_0$, we may rewrite the equation of motion in the following form,

$$ma + b(v - v_0) + qg_0 = 0,$$
$$bv_0 = qg_0, \qquad t = t_0 \Rightarrow$$
$$ma + bv - bv_0 + qg_0 = 0 \Rightarrow$$
$$ma + bv = 0$$

This way the gravitational term disappears all together from the equation of motion, and the equation is reduced to that of an object accelerating on its own power. But the gravitational term is still there, although hidden, within the equation.

This may be seen as the simplest case where the equivalence is either expressed or implied. This is the case of the rocket- in- space, which we have already examined. Another case, similar to the ship- in- the- sea example, arises if we consider that true motion in spacetime is fundamentally oscillatory.

Thus we come to the case of the simple harmonic oscillator (without damping, $b=0$), with an equation of motion

$$ma + ky = qg,$$

where

$$a = \frac{q}{m} g - \frac{k}{m} y,$$

and

$$m = q \frac{g}{a} - k \frac{y}{a}$$

Similar considerations can be made for the related parameters, as we previously did for damped motion, if we replace the damping force *bv* by the elastic force *ky*, so that

$$m = q \frac{g}{a} - k \frac{y}{a}$$

$$a = g \Rightarrow$$

$$m = q - k \frac{y}{a}$$

and

$$a = \frac{q}{m} g - \frac{k}{m} y$$

$$m = q \Rightarrow$$

$$a = g - \frac{k}{m} y$$

Choosing the necessary boundary conditions, and supposing that the acceleration of gravity is constant, $g \equiv g_0$, from the equation of motion,

$$ma + ky = qg,$$

we have,

$$t = 0, \qquad a = a_0, \qquad y = y_0, \qquad g = g_0 \Rightarrow$$

$$ky_0 = qg_0,$$

$$t = t_0, \qquad a = 0, \qquad y = 0, \qquad g = g_0$$

$$ma_0 = qg_0,$$

where

$ky_0 = ma_0 \Rightarrow$

$k = m\dfrac{a_0}{y_0} = m\omega_0^2$

Writing this equation of motion in the form,

$ma + k(y - y_0) + qg_0 = 0,$

$ky_0 = qg_0, \qquad t = 0 \Rightarrow$

$ma + ky - ky_0 + qg_0 = 0 \Rightarrow$

$ma + ky = 0$

we take the equation of motion of the simple harmonic oscillator.

The more general case can be examined if we combine the damping term and the elastic term (as well as the implicit gravitational term) into the same equation of motion,

$ma + bv + ky = qg$

$ma + bv + ky = F_0 \cos \omega t$

$F(t) = F_0 \cos \omega t = qg(t) = qg_0 \cos \omega t\,, \qquad g(t) = g_0 \cos \omega t\,, \qquad F_0 = qg_0,$

The acceleration of gravity *g(t)* can be kept constant assuming, for example, the following boundary conditions, as we have seen in the section about the forced and damped harmonic oscillator,

$t = 0, \qquad \omega = 0, \qquad y = y_0, \qquad v = 0, \qquad a = 0, \qquad F = F_0, \qquad g = g_0 \Rightarrow$

$F_0 = ky_0 = qg_0$

$t = T_0, \qquad \omega = \omega_0, \qquad y = 0, \qquad v = v_0, \qquad a = 0, \qquad F = F_0\,, \qquad g = g_0 \Rightarrow$

$F_0 = bv_0 = qg_0,$

where

$ky_0 = bv_0 = qg_0,$

$\dfrac{k}{m}y_0 = \dfrac{b}{m}v_0 = \dfrac{q}{m}g_0,$

$\omega_0^2 y_0 = \gamma v_0 = \dfrac{q}{m}g_0$

Incidentally, such an equation can be used to measure the ratio q/m.

Taking now the equation of the forced and damped harmonic oscillator,

$$ma + bv + ky = qg = F$$

and rewriting it in the following form,

$$ma + b(v - v_0) + k(y - y_0) + qg_0 = 0$$

taking also into account the boundary conditions,

$$F_0 = ky_0 = bv_0 = qg_0,$$

we have that,

$$ma + bv - bv_0 + ky - ky_0 + qg_0 = 0 \Rightarrow$$
$$ma + bv - qg_0 + ky - qg_0 + qg_0 = 0 \Rightarrow$$
$$ma + bv - qg_0 + ky = 0 \Rightarrow$$
$$ma + bv + ky = qg_0$$

In that form, the gravitational term becomes explicit, and can be identified with the driving force.

Finally, a general relationship between the inertial mass m and the gravitational charge q can be found, if we take the energy equation of the brachistochrone

$$E_B = \frac{1}{2\pi} mgL_B = M_B c^2 = mv^2$$

and replace in the gravitational term the mass m by the gravitational charge q,

$$\frac{1}{2\pi} mgL_B \rightarrow \frac{1}{2\pi} qgL_B, \qquad m \neq q \Rightarrow$$
$$E_B = \frac{1}{2\pi} qgL_B = M_B c^2 = mv^2$$

where for the mass M_B of the brachistochrone we have that,

$$g = \frac{GM_B}{R_B^2} \Rightarrow$$

$$M_B = \frac{gR_B^2}{G},$$

for the inertial mass *m* of the object moving on the brachistochrone, we have

$$qgR_B = q\frac{GM_B}{R_B} = M_B c^2 = mv^2 \Rightarrow$$

$$m = M_B \frac{c^2}{v^2} = \frac{gR_B^2}{G}\frac{c^2}{v^2} = qgR_B \frac{1}{v^2} = q\frac{GM_B}{R_B}\frac{1}{v^2},$$

while for the gravitational charge of the same object we take,

$$\frac{1}{2\pi} qgL_B = 2\pi q \frac{GM_B}{L_B} = q\frac{GM_B}{R_B} = M_B c^2 \Rightarrow$$

$$q = \frac{1}{2\pi}\frac{c^2}{G}L_B = \frac{c^2}{G}R_B$$

Comparing the ratio of the gravitational charge *q* to the inertial mass *m*, we see that

$$\frac{q}{m} = \frac{v^2}{M_B G}R_B = \frac{v^2}{gR_B},$$

while the speed *v* of the object and the speed of light *c* will be given by the following formulas,

$$v^2 = \frac{q}{m}gR_B = \frac{q}{m}\frac{GM_B}{R_B},$$

$$c^2 = \frac{qG}{R_B} = \frac{qg}{M_B}R_B$$

Thus, the choice whether we equate the inertial mass *m* to the gravitational charge *q* or not, depends on how deep we want to delve into the hidden aspects of the problem.

Faster than light

A way to support the idea of faster than light travel is the following one. In the original description of the problem of wave-particle duality, as we have already mentioned in that section, if we define the momenta of a wave and of a particle, respectively, as

$$p_m = mv$$

$$p_\mu = \frac{h}{\lambda}$$

we take an energy of the form

$$E = \frac{hc}{\lambda} = pc = mvc$$

If in this energy equation we set $v=c$, we take

$$E = mc^2, \qquad v = c$$

This energy (which is famous thanks to Einstein) treats the mass m of the system as one and the same.

If however we introduce a mass μ for the wave, then the previous relationships take the form

$$p_m = mv, \qquad \varepsilon_m = mv^2$$

$$p_\mu = \frac{h}{\lambda} = \mu c, \qquad \varepsilon_\mu = \mu c^2$$

$$\varepsilon = \frac{1}{2}mv^2 + \mu c^2 = \mu_0 c^2 = mv_0^2$$

$$\varepsilon_0 = mv_0^2 = \mu_0 c^2 = \frac{hc}{\lambda_0}$$

where the index '*0*,' in ε_0, refers to the total energy.

This analysis also reveals the aspect that we refer to an energy per wavelength λ. The notion of the brachistochrone makes clearer that the wavelength λ refers to a division of the length L of the brachistochrone, so that, by analogy, the mass μ of the wave per wavelength will refer to a division of the total mass M of the brachistochrone.

This way, while the masses μ and M refer to the oscillations of a wave of spacetime (represented by the brachistochrone), the mass m (commonly referred to as inertial mass) refers to an object travelling on the brachistochrone, whereas the wavelength λ refers to oscillations of spacetime, perceived as photons (thus it is not the Compton wavelength of the moving object).

Therefore by separating the two masses m and μ, the equation

$$\varepsilon = mv^2 = \mu c^2,$$

$$v = c \Rightarrow$$

$$\varepsilon = mc^2 = \mu c^2,$$

$$m = \mu$$

will imply an object of mass m comparable to the mass μ of the brachistochrone per wavelength, travelling at the speed of light.

The energy E stored in a region of spacetime for all wavelengths is equal to

$$E = Mc^2$$

where M is the mass equivalent of the energy E, according to Einstein's formula.

This amount of energy can also be seen as the Planck energy of the oscillations of spacetime in the same region, because anything which contains energy vibrates. If λ is the wavelength of such oscillations then the Planck energy related to those oscillations will be

$$\varepsilon = \frac{hc}{\lambda} = \mu c^2$$

This energy is supposed to be associated with photon radiation, and it is an amount of energy per wavelength λ of the photon, or per photon. The quantity μ refers to the mass equivalent of the energy ε (the energy E per wavelength λ).

If L is the linear dimension of the region of spacetime under concern, and it is composed of a number of N photons of wavelength λ, then the total energy stored in that region will be

$$E = N\varepsilon = N\frac{hc}{\lambda} = N\mu c^2 = Mc^2 \Rightarrow$$

$$E = \frac{L}{\lambda}\frac{hc}{\lambda} = \frac{L}{\lambda}\mu c^2 = Mc^2 \Rightarrow$$

$$E = \frac{hc}{\lambda^2} = \frac{L}{\lambda}\mu c^2 = Mc^2,$$

where

$$N = \frac{L}{\lambda}, \qquad M = \frac{L}{\lambda}\mu$$

The main assumption is that an object which moves in the same region of spacetime, causes spacetime to oscillate faster, or the wavelength of the photons by which those oscillations are perceived gets shorter. The increase of the frequency, or the decrease of the wavelength, of the photons is associated with the harmonics n of the oscillations. If λ_0 is the initial wavelength of the photons, and λ_n is the wavelength at some harmonic n, then it will be

$$\lambda_n = \frac{1}{n}\lambda_0, \qquad n = \frac{\lambda_0}{\lambda_n}$$

The harmonics n, which express how many times the wavelength λ_n of the photons gets shorter with respect to the original wavelength λ_0, and the number N (*or* N_n), which refers to the amount of photons which comprise the distance $L \equiv L_0$ at some harmonic n, as we have already seen, are not necessarily equal to each other:

$$n = \frac{\lambda_0}{\lambda_n}, \qquad N = \frac{L_0}{\lambda_n}, \qquad \frac{n}{N} = \frac{\lambda_0}{\lambda_n}\frac{\lambda_n}{L_0} = \frac{\lambda_0}{L_0} = \frac{1}{N_0} \Rightarrow$$

$$N_0 = \frac{N}{n}$$

where N_0 refers to the number of photons which comprise the distance L_0 at the first harmonic ($n=1$).

Therefore, if we call ε_0 the energy stored in spacetime per the initial wavelength λ_0 of the photons at the first harmonic, $n=1$,

$$\varepsilon_0 = \frac{hc}{\lambda_0} = \mu_0 c^2, \qquad n = 1,$$

and if we call ε_n the energy per the wavelength λ_n of the photons at some harmonic n,

$$\varepsilon_n = \frac{hc}{\lambda_n} = n\frac{hc}{\lambda_0} = n\mu_0 c^2 = n\varepsilon_0 = \mu_n c^2, \qquad n = n$$

then the total energy E_0, corresponding to the first harmonic, will be

$$E_0 = N_0\varepsilon_0 = N_0\mu_0 c^2 = N_0\frac{hc}{\lambda_0} = \frac{L_0}{\lambda_0}\frac{hc}{\lambda_0} = \frac{hc}{\lambda_0^2}L_0 = \frac{L_0}{\lambda_0}\mu_0 c^2 = M_0 c^2, \qquad n = 1$$

while the total energy E_n, corresponding to the harmonic n, will be

$$E_n = N\varepsilon_n = N\mu_n c^2 = N\frac{hc}{\lambda_n} = \frac{L_0}{\lambda_n}\frac{hc}{\lambda_n} = \frac{hc}{\lambda_n^2}L_0 = \frac{L_0}{\lambda_n}\mu_n c^2 \Rightarrow$$

$$E_n = \frac{hc}{\lambda_n^2}L_0 = n^2\frac{hc}{\lambda_0^2}L_0 = n^2 E_0 = n^2 M_0 c^2 = M_n c^2,$$

where

$$M_n = n^2 M_0 = n^2\frac{L_0}{\lambda_0}\mu_0 = n\frac{L_0}{\lambda_0}\mu_n = \frac{L_0}{\lambda_n}\mu_n$$

Such is the relationship between the wavelength λ_n of the photons and the distance L_0, or between the mass μ_n per wavelength λ_n and the total mass M_n, referring to a region of spacetime.

On the other hand, we have the moving object in the same region of spacetime. If this object's mass is m, and its speed is v, then its kinetic energy will commonly be

$$E_k = \frac{1}{2}mv^2$$

Since it was assumed that the motion of the object disturbs the oscillating spacetime, and that this disturbance leads to the change of the wavelength of the oscillations of spacetime, we can relate the kinetic energy of the moving object to the energy of the oscillations, which we may call E_M,

$$E_M = \frac{hc}{\lambda^2} L_0 = Mc^2$$

$$E_k = mv^2$$

The factor of ½ in front of the kinetic energy can be dropped assuming, for example, that the total kinetic energy is twice the average kinetic energy. As long as we refer to the total energy, the two forms of energy can be equated, in the sense that one form transforms into the other form, so that, using for the total energy the same symbol E, it will be

$$E = \frac{hc}{\lambda^2} L_0 = Mc^2 = mv^2$$

A first indication that the speed of the object can be greater than the speed of light is that, logically, the mass M stored in a region of spacetime will be larger than the mass m of the object moving in the same region,

$$Mc^2 = mv^2 \Rightarrow$$

$$v^2 = \frac{M}{m} c^2,$$

$$M \gg m \Rightarrow v \gg c$$

A second remark is that the wavelength λ of the oscillations of spacetime, perceived in the form of photons by an observer on board the moving object, will contract,

$$\frac{hc}{\lambda^2} L_0 = mv^2 \Rightarrow$$

$$\lambda^2 = \frac{hc}{mv^2} L_0$$

The greater the speed of the object is, the smaller the wavelength of the photons will become. Such an aspect can also be seen as a consequence of the fact that the object takes energy from spacetime, which transforms into its own kinetic energy.

Considering the harmonic n, or the number N, of the photons, the general relationship between the mass m of the object (which we may also call m_0, with reference to the inertial mass) and the mass M (or the mass μ per wavelength) stored in spacetime, can be taken as follows,

$$E_n = \frac{hc}{\lambda_n^2} L_0 = M_n c^2 = m_0 v_n^2 = n^2 E_0 = n^2 M_0 c^2 = n^2 \frac{L_0}{\lambda_0} \mu_0 c^2 = n \frac{L_0}{\lambda_0} \mu_n c^2 = \frac{\lambda_0}{\lambda_n^2} L_0 \mu_0 c^2,$$

$$n = \frac{\lambda_0}{\lambda_n}, \qquad N = \frac{L_0}{\lambda_n}, \qquad N_0 = \frac{L_0}{\lambda_0},$$

where

$$M_n = m_0 \frac{v_n^2}{c^2} = n^2 M_0 = n^2 \frac{L_0}{\lambda_0} \mu_0 = n \frac{L_0}{\lambda_0} \mu_n = \frac{\lambda_0}{\lambda_n^2} L_0 \mu_0,$$

$$M_0 = m_0 \frac{v_0^2}{c^2} = \frac{L_0}{\lambda_0} \mu_0, \qquad n = 1$$

so that if, for example, we assume that the inertial mass m_0 of the object is comparable to the mass M_0 of spacetime at the first harmonic ($n=1$), then for the speed of the object we have that

$$m_0 \frac{v_n^2}{c^2} = n^2 M_0,$$

$$m_0 \approx M_0 \Rightarrow$$

$$v_n = nc,$$

$$v_0 = c, \qquad n = 1$$

If we additionally suppose that the distance L_0 is comparable to the photon's initial wavelength λ_0, then it will also be

$$M_0 = m_0 = \frac{L_0}{\lambda_0} \mu_0, \qquad v_0 = c,$$

$$M_0 = m_0 = \mu_0, \qquad L_0 = \lambda_0$$

Significantly, in any case, the total energy E_n can be expressed solely in relation to the harmonic n of the photons, and it will be n times squared the energy of the first harmonic ($n=1$):

$$E_n = \frac{hc}{\lambda_n^2} L_0 = M_n c^2 = m_0 v_n^2 = n^2 E_0,$$

$$E_0 = \frac{hc}{\lambda_0^2} L_0 = M_0 c^2 = m_0 v_0^2 = \frac{L_0}{\lambda_0} \mu_0 c^2,$$
$$E_n = n^2 E_0 \Rightarrow$$
$$m_0 v_n^2 = n^2 m_0 v_0^2 \Rightarrow$$
$$v_n = n v_0,$$

where the condition

$$M_0 = m_0 \Leftrightarrow v_0 = c$$

is sufficient so that the final speed of the object at any harmonic *n* can be expressed solely with respect to the number *n*,

$$v_n = nc,$$
$$M_0 \approx m_0, \qquad v_0 = c$$

Notes:

Now let's attempt to put some values in the previous equations. Let's suppose, for example, that we have a spaceship of mass *m* comparable to that of a modern supercarrier, so that

$$m = 100{,}000\ tons = 1 \times 10^8 kg \approx 10^8 kg$$

Let this spaceship have to travel a distance *L* of *1ly*,

$$L = 1ly = 9.461 \times 10^{15} m \approx 10^{16} m$$

Here we will divide the distance *L* into Planck lengths l_P (instead of wavelengths λ of some reference photon), so that

$$l_P = 1.616 \times 10^{-35} m \approx 10^{-35} m$$
$$N_P = \frac{L}{l_P} \approx \frac{10^{16} m}{10^{-35} m} \approx 10^{51}$$

This region of space contains a mass equivalent *M* of N_P Planck masses m_P,

$$m_P = 2.176 \times 10^{-8} kg \approx 10^{-8} kg$$

$$M = N_P m_P = \frac{L}{l_P} m_P \approx 10^{51}(10^{-8}kg) = 10^{43}kg$$

The ratio between the mass *M* of spacetime and the mass *m* of the spaceship is

$$\frac{M}{m} \approx \frac{10^{43}kg}{10^{8}kg} = 10^{35}$$

If the spaceship consumes all this amount of available mass *M*, then its final kinetic energy will be

$$E = mv^2 = Mc^2$$

so that its final speed will be

$$v^2 = \frac{M}{m}c^2 = 10^{35}c^2 \Rightarrow$$

$$v \approx 10^{17}c$$

This is presumably the maximum possible speed for that spaceship.

But what about the energy of the reference photon? Here we will make use of the notion of the cosmic microwave background radiation (CMBR). According to Wikipedia, the photon energy of CMB photons is about 6.627×10^{-4}eV.

[https://en.wikipedia.org/wiki/Cosmic_microwave_background]

Changing the units into Joules, we have

$$1eV = 1.602 \times 10^{-19}J$$

$$\varepsilon_0 = (6.627 \times 10^{-4}eV)\left(1.602 \times 10^{-19}\ \frac{J}{eV}\right) = 1.062 \times 10^{-22}J \approx 10^{-22}J$$

This energy supposedly will be an energy covering spacetime at the ground state, *n=1*, and will refer to the energy of the CMB photons per wavelength λ_0, where

$$\varepsilon_0 = \frac{hc}{\lambda_0} \Rightarrow$$

$$\lambda_0 = \frac{hc}{\varepsilon_0} = \frac{\left(6.626 \times 10^{-34} \frac{m^2 kg}{s}\right)\left(3 \times 10^8 \frac{m}{s}\right)}{1.062 \times 10^{-22} J} = 1.872 \times 10^{-19} m \approx 10^{-19} m$$

The mass equivalent of this energy is

$$\varepsilon_0 = \frac{hc}{\lambda_0} = \mu_0 c^2 \Rightarrow$$

$$\mu_0 = \frac{\varepsilon_0}{c^2} \approx \frac{10^{-22} J}{10^{17} m^2/s^2} = 10^{-39} kg,$$

$$c^2 = \left(3 \times 10^8 \frac{m}{s}\right)^2 = 9 \times 10^{16} \frac{m^2}{s^2} \approx 10^{17} \frac{m^2}{s^2}$$

The ratio between the distance L and the wavelength λ_0 will be

$$N_0 = \frac{L}{\lambda_0} = \frac{1ly}{1.872 \times 10^{-19} m} = \frac{9.461 \times 10^{15} m}{1.872 \times 10^{-19} m} \approx \frac{10^{16} m}{10^{-19} m} = 10^{35}$$

Thus the total energy E_0 at the first harmonic, or state, will be

$$E_0 = N_0 \varepsilon_0 = \frac{L}{\lambda_0} \varepsilon_0 \approx 10^{35} (10^{-22} J) \approx 10^{13} J$$

corresponding to a mass

$$M_0 = \frac{E_0}{c^2} \approx \frac{10^{13} J}{10^{17} m^2/s^2} = 10^{-4} kg$$

The whole energy equation at the first state ($n=1$) is

$$E_0 = M_0 c^2 = m v_0^2 = \frac{hc}{\lambda_0^2} L = \frac{L}{\lambda_0} \mu_0 c^2 = \frac{L}{\lambda_0^2} l_P m_P c^2, \qquad n = 1$$

In order for the spaceship of mass m to reach the speed of light, it will have to consume an amount of mass equal to M_0:

$$v_n = v_0 = c, \qquad n = 1 \Rightarrow$$

$$M_0 c^2 = m c^2 \Rightarrow$$

$$m = M_0 \approx 10^{-4} kg$$

Presumably, this amount of mass will be sufficient for the spaceship to reach the speed of light.

The value of the wavelength λ of the reference photon which was used here is indicative, and it serves as a standard unit of length in order to divide the total distance L to be travelled.

In any case, the equation for the energy per wavelength,

$$\varepsilon = \frac{hc}{\lambda} = \frac{h}{\tau} = \mu c^2 = m v_\lambda^2, \qquad \lambda = c\tau \Rightarrow$$

$$v_\lambda^2 = \frac{\mu}{m} c^2$$

suggests that an object of mass m, travelling a distance λ at a time τ, can reach a speed v_λ (where here the index 'λ' means 'per wavelength λ') $\sqrt{\mu/m}$ times less than the speed of light, where μ is the mass stored in the distance λ, and which can be used by the moving object. If the object's mass m is comparable to the mass μ, then the object travelling the distance λ can reach the speed of light at exactly the time τ.

But if the object travels a macroscopic distance L which is N times bigger than the unit distance λ then the equation for the energy stored in that distance,

$$E = N\varepsilon = N\frac{hc}{\lambda} = N\mu c^2 = Nmv_\lambda^2 = Mc^2 = mv^2,$$

$$L = N\lambda, \qquad M = N\mu \Rightarrow$$

$$v^2 = Nv_\lambda^2 = N\frac{\mu}{m}c^2,$$

$$v_\lambda = c \Rightarrow$$

$$v^2 = Nc^2$$

implies that the object's final speed v will be $\sqrt{N}$ times greater than its speed v_λ 'per wavelength λ.' If the object's speed v_λ for the distance λ is equal to the speed of light, then the object's total speed for the distance L will be $\sqrt{N}$ times greater than the speed of light.

If we set the unit distance λ equal to Planck length l_P, then the mass μ stored in this distance will be Planck mass m_P. If the object's mass m is comparable to Planck mass, then that object's final speed v for the total distance L can be so many times greater than the speed of light, as the square root of the number $N_P=L/l_P$.

The point is that, given the huge amounts of energy stored in very small regions of spacetime, an object's final speed can be much greater than the speed of light.

Motion in the brachistochrone

The purpose here is to show that the energy of the brachistochrone is the energy of the (damped and driven) harmonic oscillator.

In the section about the inertial mass- gravitational charge equivalence, we have wondered if in the equation of motion of the driven and damped harmonic oscillator,

$$ma + bv + ky = F_0 \cos \omega t$$

the driving force can be equivalent to a gravitational force, of the form

$$F(t) = F_0 \cos \omega t = mg_0 \cos \omega t = mg(t), \qquad F_0 = mg_0, \qquad g(t) = g_0 \cos \omega t$$

This can be done as follows. First of all in the case of the simple harmonic oscillator we can show that the gravitational force is present, although 'hidden.' From the equation of motion of the simple harmonic oscillator we have that

$$\sum F = ma \Rightarrow$$

$$ma = k(y_0 - y) - mg_0$$

Setting as initial condition (at equilibrium)

$$y = 0, \qquad a = 0 \Rightarrow$$

$$ky_0 = mg_0$$

the equation of motion transforms into,

$$ma = ky_0 - ky - mg_0 \Rightarrow$$

$$ma = -ky, \qquad ky_0 = mg_0$$

so that the energy is

$$E_0 = \frac{1}{2}mv^2 + \frac{1}{2}ky^2 = \frac{1}{2}mv_0^2 = \frac{1}{2}ky_0^2$$

A way to make the factor of ½ disappear is to suppose, for example, that the previous total energy E_0 refers to half a wavelength of the oscillator.

Now we make a step forward to include a damping term in the previous energy equation. The equation of motion of the damped harmonic oscillator is

$$ma + ky + bv = 0$$

corresponding to the following total energy

$$E_0 = \frac{1}{2}m\left(\omega_0^2 + \frac{\gamma^2}{4}\right)y_0^2,$$

$$t = 0, \qquad y(0) = y_0, \qquad v(0) = -\frac{\gamma}{2}y_0, \qquad a(0) = \frac{\gamma^2}{4}y_0 - \omega'^2 y_0$$

where γ is the damping factor, and ω' is the frequency referring to the damping.

For simplicity, we apply the condition of critical damping, so that

$$\omega'^2 = \omega_0^2 - \frac{\gamma^2}{4},$$

$$\omega' = 0 \Rightarrow$$

$$\omega_0 = \frac{\gamma}{2}, \qquad \omega_0^2 = \frac{\gamma^2}{4} = \frac{k}{m}, \qquad v_0^2 = \frac{\gamma^2}{4}y_0^2 = \omega_0^2 y_0^2, \qquad a_0 = \frac{\gamma^2}{4}y_0$$

where ω' is the angular frequency of the damped oscillator, and ω_0 is the angular frequency of the simple (undamped) harmonic oscillator.

In the case of critical damping the total energy takes the following simple form,

$$E_0 = \frac{1}{2}m\left(\omega_0^2 + \frac{\gamma^2}{4}\right)y_0^2 = \frac{1}{2}m(\omega_0^2 + \omega_0^2)y_0^2 = \frac{1}{2}m(2\omega_0^2)y_0^2 = m\omega_0^2 y_0^2 \Rightarrow$$

$$E_0 = ky_0^2 = m\frac{\gamma^2}{4}y_0^2 = mv_0^2$$

In order to include a gravitational term in the previous energy equation, we take the equation of motion of the damped and driven harmonic oscillator,

$$ma + bv + ky = F_0 \cos \omega t\,, \qquad F(t) = F_0 \cos \omega t$$

and apply the initial condition,

$$t = 0, \quad \omega = 0, \quad y(0) = y_0, \quad v(0) = 0, \quad a(0) = 0, \quad F(0) = F_0 \Rightarrow$$

$$F_0 = ky_0 = mg_0$$

where the last term is derived from the simple harmonic oscillator.

Thus the total energy *E₀* is given by the following equivalent terms,

$$E_0 = mv_0^2 = ky_0^2 = m\frac{\gamma^2}{4}y_0^2 = mg_0y_0$$

Comparing this energy to the energy of the brachistochrone,

$$E_0 = mg_0R_0 = M_0c^2 = mv_0^2 = \frac{hc}{\lambda_0^2}L_0 = \frac{L_0}{\lambda_0}\mu_0c^2$$

we see that it is the same energy, if we identify the amplitude y_0 of the oscillator with the radius R_0 of the brachistochrone, so that,

$$y_0 \equiv R_0 \Rightarrow$$

$$E_0 = mg_0R_0 = M_0c^2 = mv_0^2 = kR_0^2 = m\frac{\gamma^2}{4}R_0^2 = 2\pi\frac{hc}{\lambda_0^2}R_0 = 2\pi\frac{R_0}{\lambda_0}\mu_0c^2,$$

$$E_0 = \frac{1}{2\pi}mg_0L_0 = M_0c^2 = mv_0^2 = \frac{1}{4\pi^2}kL_0^2 = \frac{1}{4\pi^2}m\frac{\gamma^2}{4}L_0^2 = \frac{hc}{\lambda_0^2}L_0 = \frac{L_0}{\lambda_0}\mu_0c^2,$$

$$R_0 = \frac{1}{2\pi}L_0$$

Also, if we have critical damping, then $v_0=c$, so that the speed of the object moving on the brachistochrone at the first state ($n=1$) can be substituted by the speed of light,

$$v_0 = c \Rightarrow$$

$$M_0 = m, \quad \frac{M_0}{\mu_0} = \frac{m}{\mu_0} = \frac{L_0}{\lambda_0} = N_0$$

The point is that since the energy of the brachistochrone is the same with the energy of the damped and forced harmonic oscillator, then the equations of motion which apply to the oscillator will also apply to the brachistochrone.

Notes:

With respect to the harmonic (elastic) term kR_0^2, and the damping term $m(\gamma^2/4)R_0^2$, which appear in the energy equation of the brachistochrone,

$$E_0 = \frac{1}{2\pi} m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{1}{4\pi^2} k L_0^2 = \frac{1}{4\pi^2} m_0 \frac{\gamma^2}{4} L_0^2 = \frac{hc}{\lambda_0^2} L_0 = \frac{L_0}{\lambda_0} \mu_0 c^2, \qquad n = 1,$$

we can define the harmonic constant k, and the damping factor γ, in such a way that, from the energy equation of the brachistochrone, at any state n,

$$E_n = n^2 E_0 = \frac{1}{2\pi} m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{1}{4\pi^2} k_n L_0^2 = \frac{1}{4\pi^2} m_0 \frac{\gamma_n^2}{4} L_0^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{L_0}{\lambda_n} \mu_n c^2$$

it will be

$$\frac{1}{4\pi^2} k_n L_0^2 = \frac{hc}{\lambda_n^2} L_0 \Rightarrow$$

$$k_n = 4\pi^2 \frac{hc}{\lambda_n^2} \frac{1}{L_0} = 2\pi \frac{hc}{\lambda_n^2} \frac{1}{R_0} = n^2 k_0,$$

$$k_0 = 4\pi^2 \frac{hc}{\lambda_0^2} \frac{1}{L_0} = 2\pi \frac{hc}{\lambda_0^2} \frac{1}{R_0},$$

where also

$$k_0 = m_0 \omega_0^2 \ \Rightarrow$$

$$k_n = n^2 k_0 = n^2 m_0 \omega_0^2 = m_0 \omega_n^2, \qquad \omega_n = n\omega_0,$$

since

$$m_0 v_n^2 = \frac{1}{4\pi^2} k_n L_0^2 \Rightarrow$$

$$k_n = 4\pi^2 \frac{m_0 v_n^2}{L_0^2} = m_0 \omega_n^2, \qquad \omega_n = 2\pi \frac{v_n}{L_0} = \frac{v_n}{R_0},$$

$$k_0 = 4\pi^2 \frac{m_0 v_0^2}{L_0^2} = m_0 \omega_0^2, \qquad \omega_0 = 2\pi \frac{v_0}{L_0} = \frac{v_0}{R_0}$$

Besides the aspect that the constant k_n here depends on the state of the system n, its particularity is that it relates the microscopic aspects (the wavelength λ of the reference photon) to the macroscopic aspects (the radius R of the brachistochrone) of the system. Therefore it is something more than a 'spring constant.'

Additionally, an approach to the damping factor γ can be made, if from the energy equation of the brachistochrone,

$$E_0 = \frac{1}{2\pi} m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{1}{4\pi^2} k_0 L_0^2 = \frac{1}{4\pi^2} m_0 \frac{\gamma_0^2}{4} L_0^2 = \frac{hc}{\lambda_0^2} L_0 = \frac{L_0}{\lambda_0} \mu_0 c^2, \qquad n = 1$$

$$E_n = n^2 E_0 = \frac{1}{2\pi} m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{1}{4\pi^2} k_n L_0^2 = \frac{1}{4\pi^2} m_0 \frac{\gamma_n^2}{4} L_0^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{L_0}{\lambda_n} \mu_n c^2$$

we use the terms

$$m_0 v_n^2 = \frac{1}{4\pi^2} m_0 \frac{\gamma_n^2}{4} L_0^2 = m_0 \frac{\gamma_n^2}{4} R_0^2, \qquad L_0 = 2\pi R_0$$

so that, it will be

$$\gamma_n = 2\frac{v_n}{R_0} = 4\pi \frac{v_n}{L_0} = n\gamma_0, \qquad \gamma_0 = 2\frac{v_0}{R_0} = 4\pi \frac{v_0}{L_0}$$

Having the condition of critical damping, $\gamma_0=2\omega_0$, $v_0=c$, we also have that

$$T_0 = \sqrt{2\pi \frac{L_0}{g_0}}$$

$$T_n = \sqrt{2\pi \frac{L_0}{g_n}} = \sqrt{2\pi \frac{1}{n^2}\frac{L_0}{g_0}} = \frac{1}{n}\sqrt{2\pi \frac{L_0}{g_0}} = \frac{1}{n} T_0,$$

$$g_n = \frac{v_n^2}{R_0} = n^2 \frac{v_0^2}{R_0} = n^2 g_0, \qquad g_0 = \frac{v_0^2}{R_0}$$

$$L_0 = v_n T_n = (n v_0)\left(\frac{1}{n} T_0\right) = v_0 T_0 = c T_0, \qquad v_0 = c$$

Thus, for the damping factor γ_n we take

$$\gamma_n = 2\frac{v_n}{R_0} = 2n\frac{v_0}{R_0} \equiv 2n\frac{c}{R_0} \equiv 2n\frac{c}{y_0} = 4\pi n\frac{c}{\lambda_0} = 2n\omega_0 = 2\omega_n,$$

$$\gamma_0 = 2\frac{v_0}{R_0} \equiv 2\frac{c}{R_0} \equiv 2\frac{c}{y_0} = 4\pi\frac{c}{\lambda_0} = 2\omega_0,$$

where,

$$v_n = nv_0 = nc, \qquad v_0 = c$$

$$R_0 \equiv y_0, \qquad y_0 = \frac{1}{2\pi}\lambda_0, \qquad \omega_0 = \frac{2\pi c}{\lambda_0}, \qquad \omega_n = n\omega_0 = \frac{2\pi c}{\lambda_n},$$

$$v_n = \omega_n R_0 = n\omega_0 R_0 = nc, \qquad c = \omega_0 R_0$$

Thus we can define the following products,

$$\gamma_0 T_0 = \gamma_0\frac{L_0}{v_0} = \left(4\pi\frac{v_0}{L_0}\right)\left(\frac{L_0}{v_0}\right) = 4\pi,$$

$$\gamma_n T_n = (n\gamma_0)\left(\frac{1}{n}T_0\right) = \gamma_0 T_0 \Rightarrow$$

$$\gamma_n T_n = \gamma_0 T_0 = 4\pi$$

$$\gamma_n T_0 = n\gamma_0 T_0 = 4\pi n$$

Introducing a time t which comes in multiples n of the initial period T_0,

$$t_n = \frac{1}{4\pi}nT_0 = \frac{1}{4\pi}n^2 T_n, \qquad t_0 \equiv \frac{1}{4\pi}T_0$$

we take

$$\gamma_n t_n = \left(\frac{4\pi n}{T_0}\right)\left(\frac{1}{4\pi}nT_0\right) = n^2,$$

$$\gamma_0 t_n = \left(\frac{4\pi}{T_0}\right)\left(\frac{1}{4\pi}nT_0\right) = n,$$

$$n = \gamma_0 t_n = \frac{v_n}{c}$$

where the time t was defined in such a way that the radian factor 4π disappears from the previous products.

This result is revealing in the sense that the previous products give us either n, or n^2, depending on whether we use the damping factor γ at the ground state ($n=1$), or at some higher state n, respectively.

The previous products and the introduction of the time t will be helpful in order to the express the energy of the brachistochrone in an exponential form, later on.

Anthropic coincidences

We have already mentioned the numerical coincidence between Planck constant h and the graviton mass m_g,

$$[h] \equiv [m_g]$$

or between the gravitational constant G and the (inverse of) the universe's linear density ρ_U,

$$[G] \equiv \left[\frac{1}{\rho_U}\right]$$

Such coincidences can be explored by using the energy equation of the brachistochrone,

$$E_B = \frac{1}{2\pi} mgL_B = M_B c^2 = mv^2 = \frac{hc}{\lambda^2} L_B = \frac{\lambda_0}{\lambda^2} L_B \mu_0 c^2 = \frac{l_P}{\lambda^2} L_B m_P c^2 = \frac{\lambda_g}{\lambda^2} L_B m_g c^2 \Rightarrow$$

$$\mu_0 \lambda_0 = m_P l_P = m_g \lambda_g = \frac{h}{c},$$

so that, for example, if the value of the graviton's wavelength λ_g coincides with *1ly*, then

$$m_g \lambda_g = \frac{h}{c} \Rightarrow$$

$$m_g = \frac{h}{c\lambda_g},$$

$$\lambda_g \equiv 1ly \Rightarrow$$

$$[m_g] = \left[\frac{h}{c\lambda_g}\right] = [h], \qquad [c] = [\lambda_g] = 1$$

Another intriguing aspect, which we have also seen, is that by replacing the length L_B of the brachistochrone with the radius R_U of the universe, and the mass M_B of the brachistochrone with the mass M_U of the universe, then the wavelength λ of the reference photon will be equal to Planck length l_P,

$$E_B = M_B c^2 = \frac{hc}{\lambda^2} L_B = \frac{l_P}{\lambda^2} L_B m_P c^2,$$

$$M_B \equiv M_U, \qquad L_B \equiv L_U \Rightarrow$$

$$M_U c^2 = \frac{hc}{\lambda^2} L_U = \frac{l_P}{\lambda^2} L_U m_P c^2 \Rightarrow$$

$$\lambda^2 = \frac{h}{c}\frac{L_U}{M_U} = 2\pi\frac{h}{c}\frac{R_U}{M_U} = 2\pi\frac{6.626\times10^{-34}\frac{m^2kg}{s}}{3\times10^8\frac{m}{s}}\frac{1.306\times10^{26}m}{1.760\times10^{53}kg} = 1.030\times10^{-68}m^2 \Rightarrow$$

$$\lambda = 1.015\times10^{-34} \equiv l_P,$$

where,

$$L_U = 2\pi R_U,$$

$$r_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616\times10^{-35}m,$$

$$l_P = 2\pi r_P = \sqrt{2\pi\frac{hG}{c^3}} = (2\pi)1.616\times10^{-35}m = 1.015\times10^{-34}m$$

Inversely, equating in the energy equation of the brachistochrone the wavelength λ of the reference photon to Planck length l_P, we take

$$E_B = M_B c^2 = \frac{hc}{\lambda^2}L_B = \frac{l_P}{\lambda^2}L_B m_P c^2,$$

$$\lambda \equiv l_P \Rightarrow$$

$$M_B c^2 = \frac{hc}{l_P^2}L_B \equiv \frac{m_P}{l_P}L_B c^2 \Rightarrow$$

$$\frac{M_B}{L_B} \equiv \frac{M_U}{L_U} = \frac{m_P}{l_P} = \frac{h}{cl_P^2} = \frac{1}{2\pi}\frac{c^2}{G},$$

$$l_P^2 = 4\pi^2 r_P^2 = 4\pi^2\frac{\hbar G}{c^3} = 4\pi^2\frac{hG}{2\pi c^3} = 2\pi\frac{hG}{c^3} \Rightarrow$$

$$\rho_U = \frac{M_U}{R_U} = \frac{m_P}{r_P} = \frac{c^2}{G}$$

If the previous ratio gives us the speed of light (squared),

$$\rho_U G = c^2 = 1\frac{ly^2}{y^2}$$

then the following product gives us the acceleration which we have called $\mathcal{g}$,

$$\frac{\rho_U G}{\lambda_g} = \frac{M_U G}{R_U}\frac{1}{\lambda_g} = \frac{m_P G}{r_P}\frac{1}{\lambda_g} = \frac{c^2}{\lambda_g} = g = 1\frac{ly}{y^2}$$

This acceleration, which is equal to 1*ly/y*^{2}, can be identified with the acceleration of gravity g_E on the Earth's surface,

$$g = \frac{\rho_U G}{\lambda_g} = \frac{M_U G}{R_U}\frac{1}{\lambda_g} = \frac{\sigma_U G R_U}{\lambda_g} = \frac{g_U R_U}{\lambda_g} = \frac{G M_E}{R_E^2} = \frac{\rho_E G}{R_E} = \sigma_E G \equiv g_E$$

This way, in order to express the coincidences between the values of fundamental quantities, we can write down the following general identity,

$$g = \frac{\rho_E G}{R_E} = \frac{\rho_U G}{\lambda_g} = \frac{c^2}{\lambda_g} = \frac{c^3 m_g}{h} = \frac{m_P G}{r_P}\frac{1}{\lambda_g},$$

where

$$\rho_U = \frac{M_U}{R_U} = \frac{m_P}{r_P} = \frac{c^2}{G}$$

$$c^2 = g\lambda_g = \rho_U G,$$

$$m_g = \frac{gh}{c^3} = \frac{h}{c\lambda_g},$$

$$\frac{h}{c} = m_g \lambda_g = m_P l_P$$

Incidentally, the term 'anthropic' in the title refers to the anthropic principle. This is a definition of the principle:

"The cosmological principle that theories of the universe are constrained by the necessity to allow human existence."
[https://www.google.gr/search?q=anthropic+principle+definition]

A consequence of this definition is not only that we *can* understand the universe, but also *why* we understand. Thus, apart from the apparent identity between numerical values, it is the ultimate relationship between physical phenomena and consciousness. But before we enter the field of

consciousness in the universe, we will explore another coincidence, in the form of a principle which I call synchronicity.

Principle of synchronicity

The idea of synchronicity is an old one, and as far as my own influences are concerned, they are largely based on Carl Jung's ideas, although his approach to the subject was mostly psychological.

Here I will formulate the basic idea of synchronicity, in the mathematical context of this document, as a set of rules, or principles, and I will explain soon afterwards.

These are the 'rules:'

1. There is a one-to-one correspondence between physical phenomena and observation (principle of analogy).
2. The speed of light is constant (principle of relativity).
3. All accelerated objects move on brachistochrones (principle of least action).
4. For a given brachistochrone, the time of the brachistochrone is constant (principle of synchronicity).
5. The distance which an object travels at the period of a photon emitted by the object, is proportional to the distance the photon travels at the time of the brachistochrone on which the object travels (condition of simultaneity).

Here is a description of the rules:
The first two rules are the same with those in the theory of relativity. The first rule (principle of analogy), in a more physical context, means that the laws of nature are the same everywhere in the universe. This is formally stated as follows:

The principle of relativity is the requirement that the equations describing the laws of physics have the same form in all admissible frames of reference.
[https://en.wikipedia.org/wiki/Principle_of_relativity]

But the principle of analogy is a more general rule, which states that that the laws of nature are the same with the laws according to which the human mind functions. A related notion is the anthropic principle- that the universe is as we know it because we are able to know. If we understand the

deeper meaning of the previous statement, then we may also understand how consciousness arises in the universe.

With respect to the principle of least action, according to Wikipedia, this is a formal definition: The path taken by a system between times t_1 and t_2 and configurations q_1 and q_2 is the one for which the action is stationary (no change) to first order.

An easier way to understand the principle, according to the same article of Wikipedia, is the following one:
Pierre de Fermat postulated that light travels between two given points along the path of shortest time, which is known as the principle of least time, or Fermat's principle.
[https://en.wikipedia.org/wiki/Principle_of_least_action]

If this is true, and since the brachistochrone is also the tautochrone (path of least time), then all (accelerating) objects will prefer this path. Non- accelerating objects, such as photons, will keep on moving on straight lines.

With respect to the time of the brachistochrone, it is said to be constant in the sense that, given the same brachistochrone, all objects falling on that brachistochrone will reach the bottom of the brachistochrone simultaneously, even if they fall from different heights. A consequence of this is that an object falling from a sufficient height will finally exceed the speed of light. There is nothing to prevent us from supposing so, and the equations presented in this document further support such an assumption.

Comparatively, the principle of relativity and the principle of synchronicity can be given by the following two formulas respectively,

$$c = \frac{L}{T} = const.$$

$$T = \frac{L}{v} = const.$$

If L and T stand for the length and the time of the brachistochrone respectively, as long as the length L is increased, the time T will also have to increase, so that the speed of light c is kept constant. But if we treat the time T as constant then the speed v of an object will increase as long as the length L of the brachistochrone is increased.

A way to bring together the previous couple of phenomenally incompatible formulas is in relation to the fifth rule. Instead of treating the time T of the brachistochrone as constant, we may treat the length L of the brachistochrone as constant. This can be done supposing that the acceleration of gravity g across the brachistochrone increases, while the length L of the brachistochrone (the distance to be travelled) stays the same. The additional amount of acceleration g, which also expresses the acceleration of the object moving on the brachistochrone, can be provided by the excited states n of the brachistochrone. Thus for a given distance L to be traveled, we have to equivalent expressions,

$L = vT$

$L = ct$

The two times T and t are not necessarily the same, since the speed c of the photon and the speed v of the object can be different. But if we relate these two times in such a way that at the first state of the brachistochrone ($n=1$) the two times are the same, then we have

$L_0 = v_n T_n$

$L_0 = ct_0$

$t_0 = T_0, \qquad n = 1 \Rightarrow$

$L_0 = v_0 T_0 = cT_0,$

$v_0 T_0 = cT_0 \Rightarrow$

$v_0 = c$

This can be set as an initial condition. More generally, at any state n, it will be

$L_0 = v_n T_n = v_0 T_0,$

$v_n = nv_0, \qquad T_n = \frac{1}{n} T_0,$

$v_0 = c \Rightarrow$

$$v_n T_n = cT_0,$$
$$v_n = \frac{T_0}{T_n} c = nc$$

The equation

$$v_n T_n = cT_0$$

is a condition of simultaneity, and expresses mathematically the principle of synchronicity.

Another way to formulate the previous condition, with respect to the period τ of the photon, is the following one. If we identify the states n of the brachistochrone with the harmonics of the reference photon, we have that

$$n = \frac{\lambda_0}{\lambda_n}, \qquad \lambda_0 = c\tau_0, \qquad \lambda_n = c\tau_n$$

where λ_n and τ_n stand respectively for the wavelength and the period of the photon, at some state n.

If, for a given state n, we divide the length L_0 of the brachistochrone into a number n of wavelengths λ_n, then, together with the initial condition,

$$L_0 = v_n T_n = cT_0$$

we have an additional final condition

$$L_0 = n\lambda_n = nc\tau_n = n\frac{1}{n}\lambda_0 = n\frac{1}{n}c\tau_0 = \lambda_0 = c\tau_0$$

so that we take,

$$v_n T_n = v_n \tau_n = cT_0,$$
$$L_0 = \lambda_0, \qquad T_0 = \tau_0, \qquad T_n = \tau_n$$

This is a mathematical description of the fifth rule, which relates the speed v_n of the object moving on the brachistochrone at some state n, to the period τ_n of the emitted (the reference) photon at the same harmonic n.

More generally, and also more accurately, the relationship between the properties of the object (e.g. its mass m_0, and speed v_n), and the properties of the reference photon (e.g. its wavelength λ_n, and period τ_n), and also the properties of the brachistochrone (e.g. its length L_0, and mass M_n), are given with respect to the energy of the brachistochrone, as we have already described. For example, in brief, we have for the energy

$$E_0 = \frac{1}{2\pi} m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{\lambda_0^2} L_0 = \frac{L_0}{\lambda_0} \mu_0 c^2 = N_0 \mu_0 c^2,$$

$$E_n = \frac{1}{2\pi} m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{\lambda_0}{\lambda_n^2} L_0 \mu_0 c^2 = \frac{L_0}{\lambda_n} \mu_n c^2 = n^2 E_0 = N_n \mu_n c^2,$$

$$n = \frac{\lambda_0}{\lambda_n}, \qquad N_0 = \frac{L_0}{\lambda_0}, \qquad N_n = \frac{L_0}{\lambda_n} = n \frac{L_0}{\lambda_0} = n N_0$$

where the numbers n and N_n express, respectively, the harmonic of the photons, and the number of photons at any state n (if $n=1$, then $N_n=N_0$).

Given the fact that the numbers n and N are not necessarily the same, for the time of the brachistochrone we have that

$$T_0 = \sqrt{2\pi \frac{L_0}{g_0}},$$

$$T_n = \sqrt{2\pi \frac{L_0}{g_n}} = \sqrt{2\pi \frac{L_0}{n^2 g_0}} = \frac{1}{n} \sqrt{2\pi \frac{L_0}{g_0}} = \frac{1}{n} T_0, \qquad g_n = n^2 g_0,$$

where

$$L_0 = \frac{1}{2\pi} g_0 T_0^2 = \frac{1}{2\pi} g_n T_n^2,$$

$$g_0 = \frac{G M_0}{R_0^2} = \frac{v_0^2}{R_0},$$

$$g_n = \frac{GM_n}{R_0^2} = \frac{v_n^2}{R_0} = n^2 \frac{GM_0}{R_0^2} = n^2 \frac{v_0^2}{R_0} = n^2 g_0, \qquad M_n = n^2 M_0, \qquad v_n = nv_0$$

For the speed of the object we have

$$v_0 = \frac{L_0}{T_0} = \frac{1}{2\pi} g_0 T_0, \qquad v_0^2 = \frac{1}{2\pi} g_0 L_0$$

$$v_n = \frac{L_0}{T_n} = \frac{1}{2\pi} g_n T_n, \qquad v_n^2 = \frac{1}{2\pi} g_n L_0 = n^2 \frac{1}{2\pi} g_0 L_0 = n^2 v_0^2, \qquad v_n = nv_0$$

while for the speed of light, as well as its wavelength and period, we have that

$$c^2 = \frac{1}{2\pi}\frac{m_0}{M_n}\, g_n L_0 = \frac{1}{2\pi}\frac{m_0}{M_0} g_0 L_0, \qquad c = \frac{\lambda_0}{\tau_0} = \frac{\lambda_n}{\tau_n}$$

Thus we take the following ratios for the two different speeds,

$$\frac{v_n}{c} = n\frac{v_0}{c} = n\frac{L_0}{T_0}\frac{\tau_0}{\lambda_0} = \frac{L_0}{T_n}\frac{\tau_0}{\lambda_0} = \frac{L_0}{T_0}\frac{\tau_0}{\lambda_n}, \qquad \frac{v_n^2}{c^2} = n^2\frac{v_0^2}{c^2} = n^2\frac{M_0}{m_0} = \frac{M_n}{m_0}$$

$$\frac{v_0}{c} = \frac{L_0}{T_0}\frac{\tau_0}{\lambda_0} = N_0\frac{\tau_0}{T_0}, \qquad \frac{v_0^2}{c^2} = \frac{M_0}{m_0}$$

Using the constant length L_0 of the brachistochrone as a guide,

$$L_0 = \frac{1}{2\pi} g_0 T_0^2 = \frac{1}{2\pi} g_n T_n^2 = v_0 T_0 = v_n T_n = N_0\lambda_0 = N_n\lambda_n = N_0 c\tau_0 = N_n c\tau_n = ct_0$$

we take the following products, which define the relationship between the speeds and the times,

$$v_n T_n = v_0 T_0 = \frac{L_0}{\lambda_0} c\tau_0 = \frac{L_0}{\lambda_n} c\tau_n = ct_0$$

where the time $\Delta t \equiv t_0$, as we have already mentioned, refers to an external observer, not travelling on the brachistochrone.

If we now suppose that as the speed v_n of the object increases, the number N_n of photons which fit in the distance L_0 is sufficiently large so that it is comparable to the harmonic n of the photons, we have that

$$n \gg 1, \qquad N_n \to n \Rightarrow$$

$$N_n = \frac{L_0}{\lambda_n} = n\frac{L_0}{\lambda_0} = nN_0 \approx n \Rightarrow$$

$$N_0 = \frac{L_0}{\lambda_0} \approx 1 \Rightarrow$$

$$L_0 \approx \lambda_0$$

If, consequently, we identify the distance L_0 with the photon's wavelength at the first harmonic, $n=1$, then it will be

$$L_0 = v_n T_n = v_0 T_0 = \lambda_0 = n\lambda_n = c\tau_0 = nc\tau_n = ct_0 \Rightarrow$$

$$v_n T_n = v_0 T_0 = c\tau_0 = ct_0$$

If, additionally, we suppose that, as the number N_n of photons which fit in the distance L_0 approaches the harmonic n of the photons, so the time T_n of the brachistochrone approaches the period τ_n of the photons, then we have that

$$T_n \to \tau_n, \qquad N_n \to n \Rightarrow$$

$$L_0 \approx \lambda_0, \qquad T_0 \approx \tau_0$$

$$v_0 = c, \qquad v_n = nc \ \Rightarrow$$

$$v_n T_n = v_n \tau_n = cT_0 = ct_0$$

In that form, the last equation compares the period τ_n of the reference photons to the speed v_n of the object, at some higher harmonic n.

Here is another way to derive the previous product in a more general form, using the energy equation of the brachistochrone,

$$E_0 = \frac{1}{2\pi} m_0 g_0 L_0 = M_0 c^2 = m_0 v_0^2 = \frac{hc}{\lambda_0^2} L_0 = \frac{L_0}{\lambda_0} \mu_0 c^2$$

$$E_n = \frac{1}{2\pi} m_0 g_n L_0 = M_n c^2 = m_0 v_n^2 = \frac{hc}{\lambda_n^2} L_0 = \frac{\lambda_0}{\lambda_n^2} L_0 \mu_0 c^2 = \frac{\lambda_0}{\lambda_n} L_0 \mu_n c^2$$

so that comparing the pair,

$$m_0 v_0^2 = \frac{L_0}{\lambda_0} \mu_0 c^2$$

we take

$$m_0 v_0^2 \lambda_0 = L_0 \mu_0 c^2 \Rightarrow$$

$$m_0 v_0^2 c \tau_0 = v_0 T_0 \mu_0 c^2 \Rightarrow$$

$$m_0 v_0 \tau_0 = T_0 \mu_0 c \Rightarrow$$

$$v_0 \tau_0 = v_n \tau_n = \frac{\mu_0}{m_0} c T_0 = \frac{\mu_n}{m_0} c T_n,$$

where

$$n = \frac{v_n}{v_0} = \frac{\lambda_0}{\lambda_n} = \frac{\tau_0}{\tau_n} = \frac{T_0}{T_n} = \frac{\mu_n}{\mu_0},$$

$$c = \frac{\lambda_0}{\tau_0} = \frac{\lambda_n}{\tau_n}, \qquad v_n = \frac{L_0}{T_n} = n \frac{L_0}{T_0} = n v_0,$$

$$L_0 = v_0 T_0 = v_n T_n$$

$$N_0 = \frac{L_0}{\lambda_0} = \frac{M_0}{\mu_0} = \frac{m_0}{\mu_0} \frac{v_0^2}{c^2}$$

$$N_n = \frac{L_0}{\lambda_n} = n \frac{L_0}{\lambda_0} = n N_0 = \frac{M_n}{\mu_n} = \frac{m_0}{\mu_n} \frac{v_n^2}{c^2}$$

Thus the constant of analogy in the fifth rule is

$$\frac{\mu_0}{m_0} = \frac{v_0^2}{c^2} \frac{\lambda_0}{L_0} = \frac{v_0}{c} \frac{\tau_0}{T_0} = \frac{M_0}{m_0} \frac{\lambda_0}{L_0} = \frac{M_0}{m_0} \frac{c}{v_0} \frac{\tau_0}{T_0}$$

The condition of simultaneity

$$v_n \tau_n \approx c T_0$$

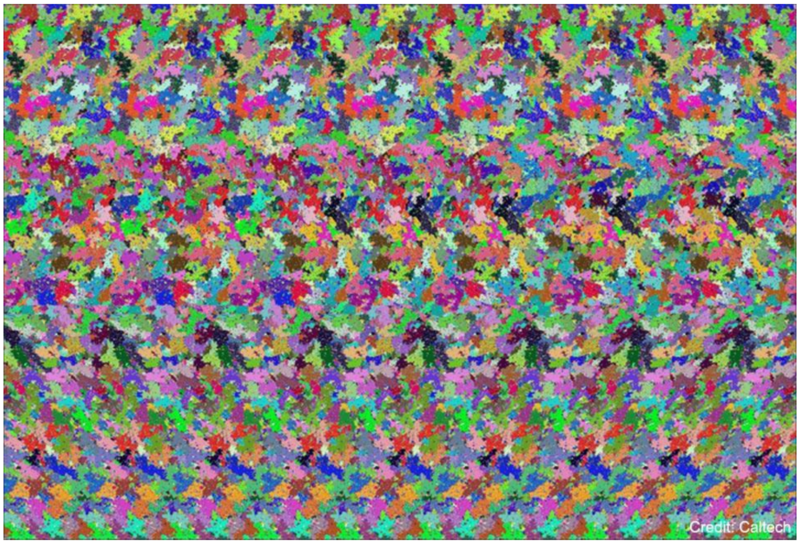
also reveals the non- local nature of synchronicity. Although the object can travel faster than light, the photon cannot. In fact the photon doesn’t need to move at all, if it is an oscillation of spacetime (in the same sense that a point on a sea wave oscillates, but it is not transferred by the wave). Such ‘points’ is what the observer measures, wherever he/she goes. Therefore his/her causal relationship with the surrounding environment is never lost.

Whether the disturbances of spacetime which are observed as photons, could also be perceived in the form of some other ‘particle,’ is another question. The point is that the principle of synchronicity sets the limits for an overall description of the problem, and can be the basis for an even more general consideration of the relationship between the observer (and his/her mind) and the phenomenon he/she observes.

A possible mathematical description of Consciousness, and its introduction into the equations, will be attempted later on.

Holographic principle

There is an interesting principle in physics, called the holographic principle. I will use the following extract as a description of the principle:



Is this picture worth a thousand words? According to the Holographic Principle, the most information you can get from this image is about $3x10^{65}$ bits for a normal sized computer monitor. The Holographic Principle, yet unproven, states that there is a maximum amount of information content held by regions adjacent to any surface. Therefore, counter-intuitively, the information content inside a room depends not on the volume of the room but on the area of the bounding walls. The principle derives from the idea that the Planck length, the length scale where quantum mechanics begins to dominate classical gravity, is one side of an area that can hold only about one bit of information. The limit was first postulated by physicist Gerard 't Hooft in 1993. It can arise from generalizations from seemingly distant speculation that the information held by a black hole is determined not by its enclosed volume but by the surface area of its event horizon. The term ‘holographic’ arises from a hologram analogy where three-dimension images are created by projecting light through a flat screen. Beware, other people looking at the featured image may not claim to see $3x10^{65}$ bits- they might claim to see a teapot.
[https://apod.nasa.gov/apod/ap170423.html]

In information theory, the information content of a system is defined as the logarithm of the states of the system. If we call the number of states *n*, then the information *I* is given as:

$$I = \log_2 n\,, \qquad [I] = bit$$
$$I = \ln n\,, \qquad [I] = nat$$
[https://en.wikipedia.org/wiki/Holographic_principle]

where the units of information (bits or nats) depend on the basis of the logarithm (base *2*, or base *e*, respectively).

Information is closely related to entropy, through Boltzmann's formula:
$$S = k_B \ln n = k_B I,$$
$$k_B = 1.381 \times 10^{-23}\,\frac{J}{K}$$

where k_B is Boltzmann's constant.

Alternatively, entropy is defined by the formula:
$$S = \frac{E}{T}$$

where *E* is the (thermal) energy and *T* is the temperature.
[https://en.wikipedia.org/wiki/Entropy]

In black hole thermodynamics, the previous formulas for the entropy take the equivalent forms,
$$S = \frac{Mc^2}{T}, \qquad E = Mc^2,$$
$$S = \frac{1}{4}k_B A, \qquad A = 4\pi R^2$$

where *S* is the entropy of the black hole, *T* its temperature, *M* its mass, *R* its radius, and *A* its area.
[https://en.wikipedia.org/wiki/Holographic_principle]

The meaning of the last formula is that since the entropy S of a black hole is analogous to its surface area A, then the information content of the black hole will be found on that area (instead of inside its volume). This is a way to express the holographic principle.

Here we will use the holographic principle in the context of the brachistochrone. First of all, the area of the brachistochrone, which we may call A_B, in relation to its radius R_B, is given by the following formula:

$$A_B = 3\pi R_B^2$$

[https://en.wikipedia.org/wiki/Cycloid#Area]

Thus, with respect to its area A_B, the energy E_B of the brachistochrone can be given as follows,

$$E_B = \frac{1}{4\pi^2} k L_B^2 = k R_B^2 = \frac{1}{3\pi} k A_B$$

The previous formula can also be written in the following form, to express the state n,

$$E_0 = k_0 R_B^2 = \frac{1}{3\pi} k_0 A_B, \qquad n = 1$$

$$E_n = n^2 E_0 = n^2 \frac{1}{3\pi} k_0 A_B = \frac{1}{3\pi} k_n A_B,$$

$$k = 4\pi^2 \frac{hc}{\lambda_n^2} \frac{1}{L_B} = 2\pi \frac{hc}{\lambda_n^2} \frac{1}{R_B} \equiv k_n, \qquad k_0 = 4\pi^2 \frac{hc}{\lambda_0^2} \frac{1}{L_B} = 2\pi \frac{hc}{\lambda_0^2} \frac{1}{R_B},$$

$$n = \frac{\lambda_0}{\lambda_n}$$

The fact that the energy E_B of the brachistochrone is proportional to its area A_B, is a good indication that the holographic principle applies to the brachistochrone.

Now we will make the assumption that the state of the brachistochrone n directly corresponds to its configuration, so that its informational content I_B, and entropy S_B (where the index '*B*' stands for '*Brachistochrone*'), will be given, respectively, by the formulas

$$I_B = \ln n$$

$$S_B = k_B I_B = k_B \ln n$$

where k_B is Boltzmann's constant (this index '*B*' stands for '*Boltzmann*').

Thus the relationship between the information I_n (or I_B), the entropy S_n (or S_B) and the energy E_n (or E_B) at any state n, will be as follows,

$$E_n = n^2 E_0 \Rightarrow$$

$$\ln E_n = \ln(n^2 E_0) = \ln n^2 + \ln E_0 = 2\ln n + \ln E_0 = 2I_n + \ln E_0\,, \qquad I_n = \ln n \Rightarrow$$

$$2I_n = \ln E_n - \ln E_0 = \ln \frac{E_n}{E_0} \Rightarrow$$

$$I_n = \frac{1}{2}\ln \frac{E_n}{E_0} \Rightarrow$$

$$S_n = k_B I_n = \frac{1}{2} k_B \ln \frac{E_n}{E_0}$$

Setting in the last equation

$$E_0 = 1, \qquad \ln E_n = \mathcal{C}$$

we take

$$2I_n = \mathcal{C} - \ln E_0,$$

$$2I_n = \mathcal{C}, \qquad \ln E_0 = 0, \qquad E_0 = 1 \ \Rightarrow$$

$$S_n = k_B I_n = \frac{1}{2} k_B \mathcal{C}$$

where

$$\mathcal{C} = \ln E_n = \ln \frac{1}{3\pi} k_n A_B,$$

$$E_n = k_n R_B^2 = \frac{1}{3\pi} k_n A_B$$

If we make a comparison between the last formula,

$$S_n = \frac{1}{2} k_B \mathcal{C} = \frac{1}{2} k_B \ln E_n = \frac{1}{2} k_B \ln \frac{1}{3\pi} k_n A_B,$$

$$E_0 = 1, \qquad \mathcal{C} = \ln E_n\,, \qquad E_n = k_n R_B^2 = \frac{1}{3\pi} k_n A_B \Rightarrow$$

$$S_B \approx \ln A_B\,, \qquad S_n \equiv S_B$$

and the formula for the entropy of black holes,

$$S = \frac{1}{4} k_B A \approx A$$

we see that we have reached a result which may be considered even more accurate, since, additionally to the fact that it gives the correct units (the units of entropy are the units of Boltzmann's constant, while the logarithm of any quantity is dimensionless), it conveys more information about the system.

The general formula

$$\ln E_B = 2I + \ln E_0 = \mathcal{C}$$

will reappear in the next section, where the meaning of the constant $\mathcal{C}$ will be identified with respect to the role of Consciousness in the universe.

Notes:

From the energy equations of the brachistochrone,

$$\varepsilon_0 = \frac{hc}{\lambda_0}$$

$$E_0 = \frac{L_B}{\lambda_0}\frac{hc}{\lambda_0} = \frac{hc}{\lambda_0^2} L_B = \frac{1}{4\pi^2} k_0 L_B^2 = k_0 R_B^2$$

$$E_n = n^2 E_0 = n^2 \frac{hc}{\lambda_0^2} L_B = \frac{hc}{\lambda_n^2} L_B = \frac{1}{4\pi^2} k_n L_B^2 = k_n R_B^2$$

$$k_n \equiv k = 4\pi^2 \frac{hc}{\lambda_n^2}\frac{1}{L_B} = 2\pi \frac{hc}{\lambda_n^2}\frac{1}{R_B}, \qquad k_0 = 4\pi^2 \frac{hc}{\lambda_0^2}\frac{1}{L_B} = 2\pi \frac{hc}{\lambda_0^2}\frac{1}{R_B}, \qquad n = \frac{\lambda_0}{\lambda_n}$$

if we replace the wavelength λ of the reference photon with Planck length l_P, we take that

$$\lambda \to l_P \Rightarrow$$

$$\varepsilon_0 = \frac{hc}{l_P} \equiv \varepsilon_P$$

$$E_B = \frac{L_B}{l_P}\varepsilon_0 \equiv \frac{L_B}{l_P}\varepsilon_P = \frac{L_B}{l_P}\frac{hc}{l_P} = \frac{hc}{l_P^2}L_B = \frac{1}{4\pi^2}kL_B^2 = kR_B^2 = n\varepsilon_P,$$

$$k = 4\pi^2\frac{hc}{l_P^2}\frac{1}{L_B} = 2\pi\frac{hc}{l_P^2}\frac{1}{R_B}, \qquad N_P = \frac{L_B}{l_P} \equiv n$$

This way the energy ε_0 per wavelength will be Planck energy ε_P, while the states n of the brachistochrone will be identified with the number N_P of Planck lengths l_P which fit in the distance L_B.

Additionally, if we set Planck energy equal to *1*, as a reference energy, then for the entropy S_B we have

$$E_B = n\varepsilon_P, \qquad n \equiv N_P = \frac{L_B}{l_P} \Rightarrow$$

$$\ln E_B = \ln(n\varepsilon_P) = \ln n + \ln\varepsilon_P = I + \ln\varepsilon_P = I,$$

$$I = \ln n, \qquad \ln\varepsilon_P = 0, \qquad \varepsilon_P = 1 \Rightarrow$$

$$S_B = k_B I = k_B \ln E_B$$

This way we arrive at a similar result for the entropy S_B in a simpler way.

Consciousness and spacetime

In the previous section we arrived at an equation of the form,

$$\ln E_B = 2I_B + \ln E_0 = \mathcal{C}$$

which connects the energy E_B (or E_n) of the brachistochrone to its informational context I_B (or I_n).

This connection is made possible by considering the states n of the system, so that the energy E_n at the n-th state will be n^2 times greater than the energy E_0 at the ground state ($n=1$),

$$E_n = n^2 E_0 \Rightarrow$$

$$\ln E_n = 2\ln n + \ln E_0 = 2I_n + \ln E_0 = \mathcal{C},$$

$$I_n = \ln n$$

Presumably the constant $\mathcal{C}$ is equal to the natural logarithm of the total energy E_n at the state n.

We can show that the previous equation is related to the energy in the exponential form,

$$E(t) = E_0 e^{-\gamma t}$$

This energy is associated with the damped harmonic oscillator, and we have already shown that the brachistochrone is an oscillator, so that the previous equation also applies to the energy of the brachistochrone.

The states n of the system can appear by setting

$$\gamma t = n \Rightarrow$$

$$E(n) = E_0 e^{-n}$$

As the number n increases, the exponential in this equation goes to $1/n^2$ (or even to $1/n$ for large enough n), so that

$$n \gg 1 \Rightarrow$$

$$E(n) = E_0 e^{-n} \approx \frac{1}{n^2} E_0 \Rightarrow$$

$$E_0 = n^2 E(n)$$

Using a negative exponential, the total energy is in fact E_0. Otherwise, if we use a positive exponential the total energy will be $E(n)$,

$E(t) = E_0 e^{\gamma t}$,

$E(n) = E_0 e^n, \qquad n = \gamma t$

$n \gg 1 \Rightarrow$

$E(n) = E_0 e^n \approx n^2 E_0 \Rightarrow$

$E(n) \equiv E_n = n^2 E_0$

Therefore either of the last two exponential forms for the energy at large n,

$E_0 = n^2 E(n), \qquad E_0 \equiv E_n$

$E(n) = n^2 E_0, \qquad E(n) \equiv E_n$

are similar to the linear form,

$\ln E_n = 2 \ln n + \ln E_0 = \ln n^2 + \ln E_0 = \ln(n^2 E_0) \Rightarrow$

$E_n = n^2 E_0$,

$\ln E_n = \mathcal{C}$

Here we will see how the constant $\mathcal{C}$ emerges in the previous equations. We can begin with the equation of motion of the damped harmonic oscillator,

$ma + ky + bv = 0$

which we can rewrite as

$ma + ky = -bv$

and multiply each side by v (as we have already done elsewhere), so that

$mav + kyv = -bv^2$

The left hand side is the change of the mechanical energy, which we can simply call E, so that

$$\frac{dE}{dt} = -bv^2$$

Comparing the rate of change of the energy E, to the energy itself, we have

$$\frac{dE}{dt} = -bv^2 = -bv^2 \frac{E}{mv^2} = -\frac{b}{m}E = -\gamma E$$

where γ is the damping factor.

If the condition of critical damping is applied, as we have already seen, the kinetic energy is equal to the elastic energy, so that the mechanical energy will be twice as much (mv^2, instead of $½mv^2$).

Solving the last differential equation, we have

$$\frac{dE(t)}{dt} = -\gamma E(t) \Rightarrow$$

$$\frac{dE(t)}{E(t)} = -\gamma dt \Rightarrow$$

$$\int \frac{dE(t)}{E(t)} = -\gamma \int dt \Rightarrow$$

$$\ln E(t) = -\gamma t + \mathcal{C} \Rightarrow$$

$$E(t) = e^{-\gamma t + \mathcal{C}} = e^{-\gamma t} e^{\mathcal{C}} = E_0 e^{-\gamma t}, \qquad E_0 = e^{\mathcal{C}}, \qquad \mathcal{C} = \ln E_0$$

The constant of integration which appears in the next to last equation

$$\mathcal{C} = \ln E(t) + \gamma t = \ln E_0$$

represents in fact the total energy E_0 of the system, for all states $\gamma t = n$, and since it is equal to the logarithm of the energy, it is a dimensionless quantity.

Formally, the information I of the system is given as the logarithm of the states,

$$I = \ln n$$

Here however, for simplicity, we will suppose that the information can be defined as the logarithm of the energy,

$$I(t) = \ln E(t)$$

In that sense, the constant $\mathcal{C}$,

$$\mathcal{C} = \ln E_0$$

will stand for the total amount of information available to the system.

This is why I call this constant $\mathcal{C}$ for *Consciousness*.

Thus the equation

$$\mathcal{C} = \ln E(t) + \gamma t = \ln E_0$$

can also be written in the equivalent form

$$\ln E_0 = \ln E(t) + \gamma t \Rightarrow$$
$$\mathcal{C} = I(t) + \gamma t$$

A more compact form can be taken by using the following substitution,

$$\ln \Gamma(t) = \gamma t = n,$$
$$\Gamma(t) = e^{\gamma t}, \qquad \Gamma^{-1}(t) = \frac{1}{\Gamma(t)} = e^{-\gamma t}$$

so that the equation for the total energy, including the factor Γ, can be written in the equivalent forms,

$$E(t) = E_0 e^{-\gamma t} = \frac{1}{\Gamma(t)} E_0 \Rightarrow$$
$$E_0 = \Gamma(t) E(t) \Rightarrow$$
$$\ln E_0 = \ln \Gamma(t) + \ln E(t) \Rightarrow$$
$$\mathcal{C} = \ln \Gamma(t) + I(t) \Rightarrow$$
$$\mathcal{C} = \gamma t + I(t),$$
$$\mathcal{C} = \ln E_0\,, \qquad I(t) = \ln E(t)\,, \qquad \Gamma(t) = e^{\gamma t}$$

Since the factor *Γ* is related to how fast information is absorbed, I will call the factor *Γ*, *Factor of Free Will*.

Incidentally, there is another reason why the letter *Γ* was used for this factor. Using the substitution,

$$\gamma t = n = \frac{v}{c}$$

we can rewrite the factor *Γ* as a function of the speed *v*,

$$\ln \mathit{\Gamma}(t) = \gamma t \Rightarrow$$

$$\ln \mathit{\Gamma}(v) = v, \qquad c = 1$$

so that the energy with respect to the speed *v* will be,

$$E(t) = E_0 e^{-\gamma t} = \frac{1}{\mathit{\Gamma}(t)} E_0$$

$$\gamma t = n = \frac{v}{c} = v, \qquad c = 1 \Rightarrow$$

$$E(v) = E_0 e^{-v} = \frac{1}{\mathit{\Gamma}(v)} E_0 \Rightarrow$$

$$E_0 = \mathit{\Gamma}(v) E(v),$$

$$\mathit{\Gamma}(v) = e^v$$

Comparing the equation for the energy in the latter form to the relativistic energy,

$$E_0 = \gamma_L(v) E(v)$$

$$\gamma_L(v) = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{1}{1 - v^2}}, \qquad c = 1$$

we see that while in relativity the energy becomes infinite at the speed of light, the energy with respect to the factor *Γ* is bounded for all speeds. This happens because while the Lorentz factor $\gamma_L(v)$ goes to infinity as the speed *v* approaches the speed of light, $v=c=1$, $c=1$, the factor $\mathit{\Gamma}(v)$ goes to infinity only when the speed *v* becomes infinite.

A final note can be made with respect to the difference in the energy,

$$E(t) = E_0 e^{-\gamma t} \Rightarrow$$

$$K(t) = \Delta E(t) = E_0 - E(t) = E_0 - E_0 e^{-\gamma t} = E_0(1 - e^{-\gamma t}) \Rightarrow$$

$$\ln K(t) = \ln \Delta E(t) = \ln[E_0(1 - e^{-\gamma t})] = \ln E_0 + \ln(1 - e^{-\gamma t}) = \ln E_0 + \ln\left[1 - \frac{1}{\Gamma(t)}\right],$$

$$\Gamma(t) = e^{\gamma t}$$

The difference ΔE in the energy can be identified with the kinetic energy (which here is called K), in the sense that the total initial energy E_0 is transformed into the motion of an object.

We may call the process by which the total potential energy E_0 of the system is transformed into the kinetic energy K of an object, '*Displacement of Consciousness*.'

In the epilogue some statements will be made about how the available energy, in the form of implicit information, transforms into the kinetic energy of a material object moving in spacetime, if this object is Consciousness itself.

Epilogue

Concluding, here is a telegraphic sketch, in relation to a possible theory of Consciousness.

Physical reality (spacetime) consists of oscillations:

$$ma + bv + ky = F_0 \cos \omega t$$

The oscillators are archetypes identified by their harmonic *n*:

$$\lambda_n = \frac{1}{n}\lambda_0$$

Each harmonic corresponds to a frequency (physical state), which can be perceived by a sense (psychic state). Experience is the collection of all harmonics.

All actions of archetypes propagate on brachistochrones (paths of least action):

$$T_n = 2\pi\sqrt{\frac{R_0}{g_n}}$$

The coordination between phenomena and perception is achieved by synchronicity:

$$v_n T_n = cT_0, \qquad v_0 = c$$

$$v_n \tau_n = cT_0, \qquad T_n \to \tau_n$$

For any harmonic or state *n*, the energy of the brachistochrone is:

$$E(n) = E_0 e^{-n}$$

The natural logarithm of the energy is the information, *I*:

$$I(n) = \ln E(n)$$

The natural logarithm of the total energy is Consciousness, $\mathcal{C}$:

$$\ln E_0 = \ln E(n) + n = \mathcal{C}, \qquad n = \gamma t$$

The natural exponential of the state or harmonic (or degree of freedom) n, is the factor of Free Will, Γ:

$\Gamma(n) = e^{n}$

The difference in the energy, is the kinetic energy, K:

$K(n) = \Delta E(n) = E_0(1 - e^{-n})$

Consciousness is displaced (comes about) by the following process:

$\ln K(n) = \ln \Delta E(n) = \ln E_0 + \ln\left[1 - \frac{1}{\Gamma(n)}\right]$

Related quantities:

$E_0: Total\ Energy$

$I_0 = \ln E_0 = \mathcal{C}: Total\ Information$

$\mathcal{C}: Constant\ of\ Consciousness$

$E(n): Remaining\ Energy$

$I(n) = \ln E\ (n): Amount\ of\ Unattained\ Knowledge$

$K(n) = \Delta E(n): Utilized\ (Kinetic)\ Energy$

$\ln K(n): Amount\ of\ Acquired\ Information$

$\Gamma(n): Factor\ of\ Free\ Will$

$\ln \Gamma(n) = \gamma t = n: Degrees\ of\ Freedom$

Related operations:

$Psychic\ content\ +\ Projection\ =\ Physical\ object$:

$E \times \Gamma = \mathcal{C}$

$Physical\ object\ +\ Reflection\ =\ Psychic\ content$:

$\mathcal{C} \times \Gamma^{-1} = E$

$Unconscious\ +\ Conscious = \ Consciousness$:

$\ln E + \ln \Gamma = \mathcal{C}$

With respect to archetypes, as both physical and psychic operators, one may read another essay of mine, called ‘Theory of the Form:’
[https://archive.org/details/TheoryOfTheForm]

------------------------- The End -------------------------

Note:

The ideas presented in this document are personal. Although these ideas are based on standard scientific knowledge, their implications may far exceed such knowledge. Incidental mistakes in the calculations may have been made.

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